PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Convection in the Presence of Restraints

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CONVECTION IN THE PRESENCE OF RESTRAINTS

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CONTENTS

In the presence of rotation or a magnetic field, the linearized convection problem reduces to a cubic characteristic equation. In part I, general methods are given for determining the onset of convection; in particular, the transition from oscillatory to steady modes is considered. The importance of this transition arises from evidence that oscillatory modes are inefficient at transporting heat. These methods are then applied to a rotating system where the critical Rayleigh number can be expressed in terms of a Taylor number. It is found that overstable modes develop into steady unstable modes before the exchange of stabilities for Prandtl numbers less than one-third. The nature of the motions is discussed and a similar treatment is provided for convection in a magnetic field.

In part II, criteria for the onset of instability are derived from physical arguments. Convection can be treated by balancing the work done by buoyancy forces against the energy dissipated. In a rotating system, the effect of the Coriolis forces is to restrict the cell width and thus to enhance dissipation and promote stability. A magnetic field similarly attenuates the cells and prevents steady convection until the liberated kinetic energy exceeds the energy in the field.

In part III, a cellular model is proposed for turbulent convection in a fluid of negligible viscosity, where the motion is limited by the non-linear transfer of energy to smaller-scale motions. If the Rayleigh number $R_3 = g\alpha\beta d^4/\pi^4\kappa^2 \gg 1$ the convective transport varies as $R_3^{\frac{1}{3}}$, while it varies as R_3^2 when $R_3 \ll 1$. The discussion is extended to convection in the presence of rotation or a magnetic field; it is shown that overstable perturbations cannot develop into steady turbulent convection unless the system is already unstable to non-oscillatory modes. The transition from overstable to steady modes should therefore correspond to a sharp increase in convective transport.

I. THE ONSET OF INSTABILITY

1. Introduction

The onset of convection in a layer of fluid heated from below was first considered by Lc
Rayleigh (1916) and, in detail, by Pellew & Southwell (1940). The effects of a magne
field were treated by Thompson (1951) who showed that overstable oscillations cou
arise; Chandrasekhar derived criteria for the exchange of stabilities and the onset of ov
stability in the presence of rotation or a magnetic field: this work can be found in his bo
(Chandrasekhar 1961) whose terminology will be followed here wherever possible. I

Table 1. Notation

Prandtl numbers

$$p_1 = \frac{\nu}{\kappa} \qquad p_2 = \frac{\nu}{\eta} \qquad p_3 = \frac{\kappa}{\eta} = \frac{p_2}{p_1}$$

Rayleigh numbers

$$R = \frac{g\alpha\beta d^4}{\kappa\nu} \qquad R_3 = p_1 R_1 = \frac{g\alpha\beta d^4}{\pi^4\kappa^2}$$

$$R_1 = \frac{R}{\pi^4} = \frac{g\alpha\beta d^4}{\pi^4\kappa\nu} \qquad R_4 = p_2 R_2 = \frac{b^2-1}{b^2} R_5$$

$$R_2 = \frac{b^2-1}{b^2} R_1 \qquad R_5 = p_2 R_1 = \frac{g\alpha\beta d^4}{\pi^4\eta\kappa}$$

Taylor numbers

$$T = \frac{4\Omega^2 d^4}{\nu^2} \qquad T_2 = \frac{T_1}{b^2}$$

$$T_1 = \frac{T}{\pi^4} = \frac{4\Omega^2 d^4}{\pi^4\nu^2} \qquad T_3 = p_1^2 T_1 = \frac{4\Omega^2 d^4}{\pi^4\kappa^2}$$

Chandrasekhar numbers

$$Q = \frac{\mu H^2 d^2}{4\pi\rho\eta\nu} \qquad Q_3 = \frac{p_1^2}{p_2} Q_1 = \frac{\mu H^2 d^2}{4\pi^3\rho\kappa^2}$$

$$Q_1 = \frac{Q}{\pi^2} = \frac{\mu H^2 d^2}{4\pi^3\rho\eta\nu} \qquad Q_4 = p_2 Q_1 = \frac{\mu H^2 d^2}{4\pi^3\rho\eta^2}$$

$$Q_2 = p_1 Q_1 = \frac{\mu H^2 d^2}{4\pi^3\rho\eta\kappa} \qquad q = Q_4^{\frac{1}{4}}$$

Characteristic times

$$\tau_\eta = \frac{d^2}{\pi^2 b^2 \eta} \qquad \tau_\kappa = \frac{d^2}{\pi^2 b^2 \kappa} \qquad \tau_\nu = \frac{d^2}{\pi^2 b^2 \nu}$$

a	half-width of convection cell
b	horizontal scale factor: $b^2 = 1 + k^2 d^2/\pi^2$ and so for a square cell: $b^2 = 1 + 2d^2/a^2$
c_p	specific heat per unit mass at constant pressure
d	layer depth
g	gravitational acceleration
k	horizontal wave number: $k^2 = \nabla^2 - \partial^2/\partial z^2$
s	time constant: $s = \frac{\sigma d^2}{\pi^2 \nu}$; $s_1 = ps = \frac{\sigma d^2}{\pi^2 \kappa}$
$\mathbf{u}$	velocity vector
w	vertical component of velocity $\mathbf{u}$
$\mathbf{B}$	magnetic induction
D	d/dz
E	energy transport (ergs cm^{-2} s^{-1})
F	energy transport: $E/c_p\rho$
N	Nusselt number: $F/\beta\kappa$
P	pressure
α	volume coefficient of thermal expansion
β	superadiabatic temperature gradient
γ	$g\alpha\beta$
ϵ	rate of transfer of energy through inertial subrange: $\epsilon = w^3/d$
ζ	vertical component of curl $\mathbf{u}$
η	resistivity: $\eta = (4\pi\mu\sigma)^{-1}$ where σ is the electrical conductivity
ϑ	temperature perturbation
κ	thermometric conductivity
ν	kinematic viscosity
ρ	density
σ	time constant
Δ	discriminant of cubic equation
Θ	temperature
Ω	angular velocity of rotation

treatment is extended here to cover the transition from oscillatory to steady modes. There is some experimental evidence that oscillations are less efficient than steady modes at transporting heat so it is also important to discover whether overstable modes can develop into unstable steady modes as the temperature gradient is increased. This possibility was first pointed out by Danielson (1961 *b*) in connexion with the magnetic fields of sunspots.

In this part a general treatment is given for the onset of convection and transition from oscillatory to steady motions; this is then applied to the onset of convection in a rotating system and—more briefly—to the effects of a magnetic field. The physical principles underlying this treatment are discussed in part II and turbulent convective transport is, finally, considered in part III.

The notation used is consistent throughout the paper and is summarized in table 1.

2. General treatment

We shall consider the onset of convection in systems with an imposed external restraint such as rotation or the presence of a magnetic field. Our method is to assume the usual Boussinesq approximation and then to linearize the equations of motion; the treatment therefore refers only to small disturbances. We can express these perturbations in terms of a set of orthogonal modes, each of which may then be treated separately.

We suppose that such a mode varies exponentially with time, as e^{st}: then the equations of motion with suitable boundary conditions will give us a characteristic equation for s. In the examples with which we shall be concerned, this will be a cubic and can be written

$$s^3+As^2+Bs+C=0, \tag{1}$$

where A, B and C can be expressed as functions of the wave number of the particular mode, of the Rayleigh number

$$R=g\alpha\beta d^4/\kappa\nu, \tag{2}$$

of the Prandtl numbers

$$p_1=\nu/\kappa \quad \text{and} \quad p_2=\nu/\eta, \tag{3}$$

and of a dimensionless number characterizing the restraint. For rotation this is the Taylor number

$$T=4\Omega^2d^4/\nu^2, \tag{4}$$

while for a magnetic field the relevant quantity is

$$Q=\mu H^2d^2/4\pi\rho\eta\nu \tag{5}$$

which I shall call the 'Chandrasekhar number' after its inventor.

Equation (1) has three solutions; of these, one is always real while the other two are either real or else complex conjugates. Convective instability will correspond to modes whose amplitude increases with time. Thus, if s is real, instability will occur when s is positive, while if s is complex instability can set in as oscillations whose amplitude increases exponentially with time (Eddington termed this 'overstability') when the real part of s is positive. So we must seek the limiting conditions for the onset of instability. If s is real we search for a state of 'marginal stability' when $s=0$. (The terminology originally developed by Poincaré for the stability of rotating masses has been applied to the problem of convection also and the transition from stability to instability without oscillations is injudiciously described as an 'exchange of stabilities'.) If s is complex, we want the condition that its real part be zero for the onset of overstability.

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From equation (1) we can immediately see that the state of marginal stability with $s = 0$ can only occur if

$$C = 0. \tag{6}$$

On the other hand, for the onset of overstability s must be purely imaginary, say

$$s = \mathrm{i}s_1,$$

so that

$$(As_1^2 - C) + \mathrm{i}s_1(s_1^2 - B) = 0, \tag{7}$$

whence, if we equate real and imaginary parts separately to zero,

$$AB - C = 0. \tag{8}$$

Note that, since $s_1^2 \geqslant 0$, (7) can only be satisfied if

$$B \geqslant 0 \quad \text{and} \quad AC \geqslant 0. \tag{9}$$

The general conditions for stability, i.e. for the real parts of the roots of (1) to be all negative, can be obtained from the Routh–Hurwitz criterion (see, for example, Uspensky 1948) and these are

$$\text{(i)} \qquad A > 0, \tag{10}$$

$$\text{(ii)} \qquad C > 0 \tag{11}$$

and

$$\text{(iii)} \qquad AB - C > 0. \tag{12}$$

By comparison with (6) and (8) we can see that (11) corresponds to the exchange of stabilities while (12) refers to the onset of overstability.

In the problems with which we shall be concerned, condition (10) will be trivially satisfied; so overstability is only possible if

$$B \geqslant 0; \quad C \geqslant 0 \tag{13}$$

from equation (9). The latter condition shows that overstability is impossible once the exchange of stabilities has occurred, while the former condition is less strong than (12), which can be written

$$B > C/A. \tag{14}$$

Thus overstability arises when

$$AB < C \tag{15}$$

provided that

$$C > 0. \tag{16}$$

If we substitute

$$x = s - \tfrac{1}{3}A, \tag{17}$$

equation (1) can be rewritten as

$$x^3 + Ex + F = 0, \tag{18}$$

where

$$E = B - \tfrac{1}{3}A^2 \tag{19}$$

and

$$F = \tfrac{2}{27}A^3 - \tfrac{1}{3}AB + C. \tag{20}$$

Then we can define the discriminant

$$\Delta = 4E^3 + 27F^2 \tag{21}$$

and the condition for the roots of (1) and (18) to be real is that

$$\Delta \leqslant 0 \tag{22}$$

(see Uspensky 1948, or Turnbull 1952).

From the inequalities (10), (11), (12) and (22) it is then possible to decide all that we require as to whether the solutions of (1) are real, complex, stable or unstable. For fixed Prandtl numbers and a given horizontal pattern (defined by the parameter b) we can divide the R–T or R–Q planes into regions where different solutions apply, as, for instance, in figure 1. We see here that for $T > T^{(o)}$ overstability can occur and that the overstable oscillations develop into steady unstable motions as the Rayleigh number is increased.

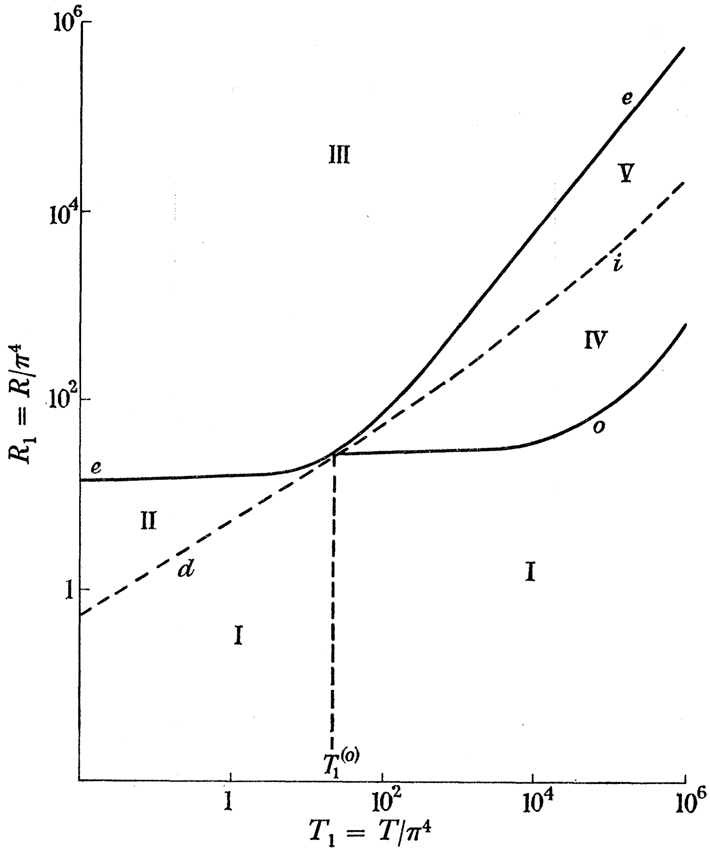

FIGURE 1. The R_1–T_1 plane, showing the onset of overstable and steady instabilities ($p_1 = 0{\cdot}025$, $b^2 = 3$). Curve o gives the onset of overstability ($AB = C$); curve di is the transition to steady modes ($\Delta = 0$) and e is the exchange of stabilities ($C = 0$). For the numbered regions see table 2.

TABLE 2. NATURE OF SOLUTIONS OF EQUATION (1) IN DIFFERENT REGIONS OF FIGURE 1

region		nature of solutions
I	1 decaying steady mode	(s real and negative)
	2 decaying oscillatory modes	(s complex, with negative real parts)
II	3 decaying steady modes	(s real and negative)
III	2 decaying steady modes	(s real and negative)
	1 unstable steady mode	(s real and positive)
IV	1 decaying steady mode	(s real and negative)
	2 overstable oscillatory modes	(s complex, with positive real parts)
V	1 decaying steady mode	(s real and negative)
	2 unstable steady modes	(s real and positive)

Let us use Danielson's terminology and refer to the curve $\Delta = 0$ as the 'onset of instability' for $T > T^{(o)}$; when $T < T^{(o)}$ let us call it the 'onset of steady modes'. After crossing the onset of instability curve there are two unstable modes and one of these becomes stable if we further increase the Rayleigh number so as to cross the exchange of stabilities curve. Thus, for $T > T^{(o)}$, points above the exchange of stabilities curve have one *less* unstable mode than points below it, contrary to what applies for $T < T^{(o)}$.

The nature of the solutions to equation (1) in the different regions of figure 1 is shown in table 2. In figure 2 the solutions s are plotted in the complex plane for fixed T and increasing R in the four cases $T < T^{(o)}$, $T = T^{(o)}$, $T > T^{(o)}$ and $T \gg T^{(o)}$. Note that there is always at least one decaying steady mode.

In the following sections the general methods already outlined will be applied to the particular cases of rotation and a magnetic field and we shall bear in mind the possible astrophysical applications of the theory.

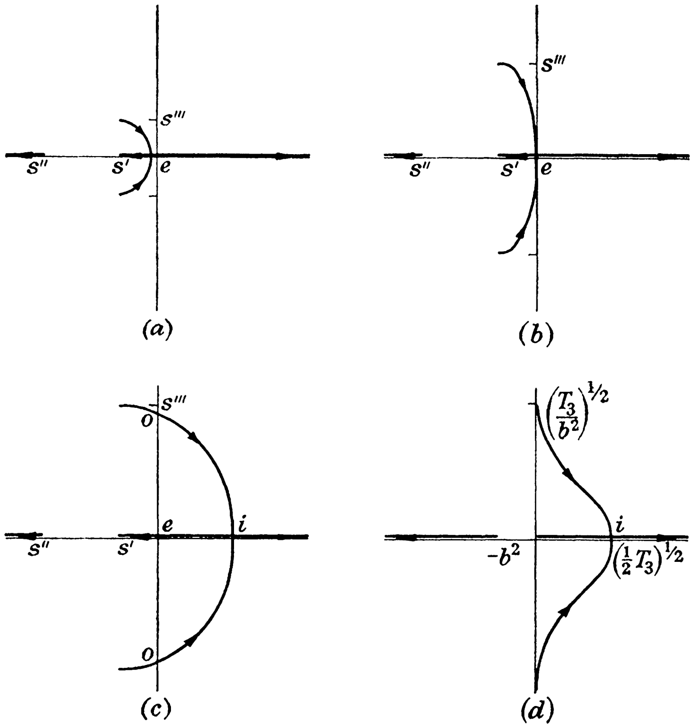

FIGURE 2. Location of solutions to the characteristic equation in the complex plane ($p \ll 1$). (*a*) Values of s as R is increased, for $T_1 \ll T_1^{(o)} = (1+p)\, b^6/(1-p)$; (*b*) the same, but for $T_1 = T_1^{(o)}$; (*c*) the same, but for $T_1^{(o)} \ll T_1 \ll b^6/p^2$; (*d*) values of $s_1 = ps$ as R is increased, for $b^6/p^2 \ll T_1$. Points labelled o, i, e denote the onset of overstability, the transition to unstable steady modes and the exchange of stabilities, respectively. $s' = -b^2$; $s'' = -b^2/p$; $s''' = i(T_1/b^2)^{\frac{1}{2}}$.

3. THE ONSET OF CONVECTION IN A ROTATING SYSTEM

(*a*) *The onset of overstability and the exchange of stabilities*

The linearized equations for convective motion in a rotating system are derived in Chandrasekhar's book; it is impossible to discuss convection without introducing a bewildering number of symbols but a guide to these is given in table 1.

Let us consider a fluid confined between two infinite horizontal planes a distance d apart, all rotating with an angular velocity Ω about the vertical z axis. On this is imposed a vertical temperature gradient β. Let α be the volume coefficient of thermal expansion, κ the thermometric conductivity, ν the kinematic viscosity and g the acceleration due to gravity. Then take small perturbations $\mathbf{u}$ and ϑ of velocity and temperature respectively and make the usual Boussinesq approximation. For simplicity let us assume free boundaries, at which only the normal component of $\mathbf{u}$ must vanish. Then we can restrict our attention to the vertical components w and ζ of the vectors $\mathbf{u}$ and $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$, and these satisfy equations of the form

$$\mathsf{L}\mathbf{X} = \partial \mathbf{X}/\partial t, \tag{23}$$

where $\mathsf{L}(\nabla_1^2, \partial/\partial z)$ is a matrix of differential operators, $\nabla_1^2 = \nabla^2 - \partial^2/\partial z^2$ is the two-dimensional Laplacian and the column vector

$$\mathbf{X} = \begin{pmatrix} w \\ \zeta \\ \vartheta \end{pmatrix}.$$

We shall restrict our attention to the class of disturbances that can be expanded in a series of orthogonal eigenvectors $\mathbf{X}^{(i)}$, varying exponentially with time; each $\mathbf{X}^{(i)}$ is a normal mode of the system and has a time constant equal to the corresponding eigenvalue σ_i. (This is a class of which each member can be defined by a single component only, so that if w is arbitrarily chosen then ζ and ϑ are automatically fixed. Without this restriction, an extension of the concept of self-adjointness, it would be impossible to use an eigenvector expansion. Mathematically, this is a significant limitation but it is not physically important since an arbitrary disturbance will develop into a member of this class in any case.)

Any such disturbance that satisfies the boundary conditions can then be expressed uniquely in terms of a complete set of orthogonal modes of which one may be written

$$\mathbf{X} = \begin{pmatrix} w \\ \zeta \\ \vartheta \end{pmatrix} = \begin{pmatrix} W \\ Z \\ \Theta \end{pmatrix} f(x, y)\, \mathrm{e}^{\sigma t}, \tag{24}$$

where W, Z and Θ are functions of z only whose relative magnitudes are fixed and $f(x, y)$ satisfies

$$\nabla_1^2 f = -k^2 f. \tag{25}$$

The choice of the function f is then determined by the boundaries and symmetry of the system but does not affect the eigenvalue problem provided (25) is satisfied. In particular, it is often convenient to take

$$f(x, y) = \exp \mathrm{i}(k_x x + k_y y); \quad k_x^2 + k_y^2 = k^2. \tag{26}$$

For convenience, we choose dimensionless units, measuring lengths in terms of the depth d of the layer and writing

$$p = \frac{\nu}{\kappa}, \quad s = \frac{d^2}{\pi^2 \nu}\sigma, \quad h = \frac{kd}{\pi}, \quad \mathrm{D} = \frac{d}{\pi}\frac{\mathrm{d}}{\mathrm{d}z}. \tag{27}$$

Equations (23) and (24) then give

$$(\mathrm{D}^2 - h^2 - s)(\mathrm{D}^2 - h^2) W = (2\Omega d^3/\pi^3 \nu)\, \mathrm{D}Z + (g\alpha d^2/\pi^2 \nu)\, h^2 \Theta,$$

$$(\mathrm{D}^2 - h^2 - s) Z = -(2\Omega d/\pi \nu)\, \mathrm{D}W,$$

$$(\mathrm{D}^2 - h^2 - ps)\, \Theta = -(\beta d^2/\pi^2 \kappa)\, W.$$

Let us finally define

$$b^2 = 1 + h^2 = 1 + k^2 d^2/\pi^2. \tag{28}$$

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The characteristic equation is then

$$s^3+As^2+Bs+C=0$$

with

$$A=(b^2/p)\,(1+2p), \tag{29}$$

$$B=(1/p)\,[b^4(2+p)+pT_2-R_2], \tag{30}$$

$$C=(b^2/p)\,[b^4+T_2-R_2], \tag{31}$$

where

$$T_2=\frac{T_1}{b^2}=\frac{T}{\pi^4 b^2} \tag{32}$$

and

$$R_2=\frac{b^2-1}{b^2}R_1=\frac{b^2-1}{\pi^4 b^2}R. \tag{33}$$

The general methods of § 2 now yield criteria for the onset of overstability and the exchange of stabilities to take place. Thence we can establish, as a function of the Taylor number, the minimum Rayleigh number for which either form of instability can occur.

The exchange of stabilities then occurs at a Rayleigh number

$$R_1^{(e)}=\frac{1}{b^2-1}[b^6+T_1], \tag{34}$$

while overstability sets in at

$$R_1^{(o)}=\frac{2(1+p)}{b^2-1}\left[b^6+\frac{p^2}{(1+p)^2}T_1\right]. \tag{35}$$

Moreover, overstability is impossible unless $R_1^{(o)}<R_1^{(e)}$, for which a necessary condition is that $p<1$. Then the lowest value of T_1 for which overstability can occur with a given b^2 is

$$T_1^{(o)}=\frac{1+p}{1-p}b^6, \tag{36}$$

when overstability sets in at

$$R_1=\frac{2}{1-p}\,\frac{b^6}{b^2-1}. \tag{37}$$

From (34), $R_1^{(e)}$ attains its minimum value of $R_{1,\,\text{min.}}^{(e)}$ with corresponding b_e^2 when

$$T_1=2b_e^6-3b_e^4$$

and so

$$R_{1,\,\text{min.}}^{(e)}=3b_e^4, \tag{38}$$

whence we can express T_1 as an explicit function of $R_{1,\,\text{min.}}^{(e)}$:

$$T_1=2(\tfrac{1}{3}R_{1,\,\text{min.}}^{(e)})^{\frac{3}{2}}-R_{1,\,\text{min.}}^{(e)} \tag{39}$$

and as $T_1\to\infty$, from (39),

$$R_{1,\,\text{min.}}^{(e)}\to 3(\tfrac{1}{2}T_1)^{\frac{2}{3}}. \tag{40}$$

Similarly, for the onset of overstability

$$R_{1,\,\text{min.}}^{(o)}=6(1+p)\,b_o^4 \tag{41}$$

and

$$T_1=\frac{(1+p)^2}{p^2}\left[2\left\{\frac{R_{1,\,\text{min.}}^{(o)}}{6(1+p)}\right\}^{\frac{3}{2}}-\left\{\frac{R_{1,\,\text{min.}}^{(o)}}{2(1+p)}\right\}\right], \tag{42}$$

so that as $T_1\to\infty$ we have

$$R_1\to 6\left\{\frac{p^4}{1+p}\right\}^{\frac{1}{3}}(\tfrac{1}{2}T_1)^{\frac{2}{3}}. \tag{43}$$

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Most of these results, which are reproduced here for completeness, have already been derived by Chandrasekhar (1961). However, the explicit expressions (39) and (42) have not been given before.

(b) *The transition from oscillatory to steady motions*

We must now investigate the transition from overstable to non-oscillatory convective modes. So we have first to obtain the discriminant Δ and to find the condition that it be negative. Because of the complexity of this expression we then consider the extremes when T_1 is very large or very small and derive the minimum Rayleigh number for the onset of steady instability in either case.

Referring to equations (19), (20) and (29) to (31) we have here

$$E = (1/3p^2)\,[-(1-p)^2 b^4 + 3p(pT_2 - R_2)] \tag{44}$$

and

$$F = (b^2/27p^3)\,(1-p)\,[2(1-p)^2 b^4 + 9p(2pT_2 + R_2)], \tag{45}$$

whence

$$\Delta = (1/p^4)\,[4(1-p)^4 T_2 b^8 + (1-p)^2 (8p^2 T_2^2 + 20pT_2 R_2 - R_2^2)\, b^4 + 4p(pT_2 - R_2)^3]. \tag{46}$$

Then for the roots of the characteristic equation to be real we require that

$$\Delta \leqslant 0$$

which becomes

$$f = 4(1-p)^4 T_1 b^{12} + (1-p)^2 [8p^2 T_1^2 + 20pT_1 R_1 (b^2-1) - R_1^2 (b^2-1)^2]\, b^6 + 4p[pT_1 - R_1(b^2-1)]^3 \leqslant 0. \tag{47}$$

A necessary, though not a sufficient, condition for this inequality to be satisfied is that

$$R_1 > pT_1/(b^2-1). \tag{48}$$

Now

$$f = 0 \tag{49}$$

is a cubic in R_1 with a solution $R_1^{(d)}(T_1, b^2)$ where $R_1^{(d)}$ is the minimum value of R_1 for which a non-oscillatory mode can occur for a given T_1 and b^2. We shall also be interested in finding the minimum value of $R_1^{(d)}$ for a given T_1; that is, we want to solve (49) subject to

$$\partial f/\partial b^2 = 0; \tag{50}$$

and from (47)

$$\partial f/\partial b^2 = 24(1-p)^4 T_1 b^{10} + (1-p)^2 [3b^4\{8p^2 T_1^2 + 20pT_1 R_1 (b^2-1) - R_1^2 (b^2-1)^2\} + b^6\{20pT_1 R_1 - 2R_1^2 (b^2-1)\}] - 12pR_1 [pT_1 - R_1(b^2-1)]^2. \tag{51}$$

The variation of $R_1^{(d)}$ with T_1 for a given value of b^2 has already been shown in figure 1. However, no purpose is served by searching for a general solution of (49) or (51), since the resulting expressions are too involved to be of any value. It suffices to evaluate $R_1^{(d)}$ when T_1 is either small (so that the first term dominates equation (47)) or very large.

(i) Let us consider first the case when T_1 is small, i.e.

$$T_1 \ll (1-p)^2 b^6/2p^2. \tag{52}$$

Then (49) yields

$$R_1^{(d)} = 2(1-p)\, b^3 T_1^{\frac{1}{2}}/(b^2-1), \tag{53}$$

which is a minimum when $b^2 = 3$; that is to say, that for a given

$$T_1 \ll 27(1-p)^2/2p^2 \tag{54}$$

steady modes first appear at

$$R_1 = R^{(d)}_{1,\,\text{min.}} = 3(1-p)\,(3T_1)^{\frac{1}{3}}, \tag{55}$$

with

$$b^2 = b^2_d = 3. \tag{56}$$

(ii) Let us now suppose that T_1 is very large and that

$$T_1 \gg (1-p)^2\, b^6/2p^2; \tag{57}$$

then, from (47) and (49),

$$R^{(d)}_1 = pT_1/(b^2-1). \tag{58}$$

This expression decreases monotonically with increasing b^2, so we must obtain a better approximation before attempting to find a minimum value. Let us put

$$R^{(d)}_1 = [pT_1 + x]/(b^2-1). \tag{59}$$

Then

$$x = 3[\tfrac{1}{4}p(1-p)^2\, T_1^2]^{\frac{1}{3}}\, b^2 \ll pT_1 \tag{60}$$

and $R^{(d)}_1$ is least when

$$(1-p)^2\, b^6/4p^2 \sim T_1. \tag{61}$$

This contradicts the assumption in (57) and so (58) cannot hold near a minimum value of $R^{(d)}_1$.

(iii) We are therefore forced to consider the situation when (61) applies. If p is small, this will only arise when T_1 is very large; let us assume that

$$p^2T_1/(1-p)^2 \gg b^2 \gg 1. \tag{62}$$

Then (47) and (51) can be written as

$$4y^3 + x(y^2-18y-27) - 4x^2 = 0$$

and

$$12y^2(y+1) + x(5y^2-70y-99) - 24x^2 = 0,$$

where

$$(1-p)^2\, b^6 = xp^2T_1,$$
$$R_1(b^2-1) = (1+y)\,pT_1.$$

Thence we find that

$$b^6_d = 2p^2T_1/(1-p)^2 \tag{63}$$

and

$$R^{(d)}_{1,\,\text{min.}} = \tfrac{9}{2}[\tfrac{1}{2}p(1-p)^2]^{\frac{1}{3}}\, T_1^{\frac{2}{3}}. \tag{64}$$

Comparison of (64) with (40) shows that

$$R^{(d)}_{1,\,\text{min.}} = \tfrac{3}{2}[2p(1-p)^2]^{\frac{1}{3}}\, R^{(e)}_{1,\,\text{min.}}. \tag{65}$$

Thus, for a given T_1 satisfying (62), $R^{(d)}_{1,\,\text{min.}}$ is zero when $p = 0$ or 1 and has a maximum value, equal to $R^{(e)}_{1,\,\text{min.}}$, when $p = \frac{1}{3}$.

(c) *Different forms of instability and the effect of varying the Prandtl number*

We have found the values of the Rayleigh number at which the transition from oscillatory to steady modes occurs; we must now establish the range over which this corresponds to a change from overstable to steadily increasing instabilities. In the range where $R^{(d)}$ corresponds to the onset of steady instabilities, we shall denote it by $R^{(i)}$.

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We can then proceed to compare the values of the Rayleigh number at which the various modes of convection will set in and to establish the relative magnitudes of $R^{(o)}_{1,\,\mathrm{min.}}$, $R^{(i)}_{1,\,\mathrm{min.}}$, and $R^{(e)}_{1,\,\mathrm{min.}}$. In particular, we want to decide under what conditions $R^{(i)}_{1,\,\mathrm{min.}} < R^{(e)}_{1,\,\mathrm{min.}}$ so that steady convection will be able to set in before the exchange of stabilities. We shall find that for $T_1 > 27$ this will always be true for small Prandtl numbers. However, if the Prandtl number is increased, this range is diminished and when $p > \frac{1}{3}$ the exchange of stabilities will always occur first.

(i) When the Taylor number is sufficiently small for (52) to be satisfied, $R^{(d)}_1$ is given by (53) and corresponds to the transition from oscillatory to steady instabilities if

$$p \ll 1. \tag{66}$$

Then this transition will first occur with

$$T_1 = b^6 \quad \text{and} \quad R_1 = 2b^6/(b^2-1) \tag{67}$$

so that for

$$p \ll 1 \quad \text{and} \quad b^6 \ll T_1 \ll b^6/p^2, \tag{68}$$

$$R^{(i)}_1 = 2b^3 T_1^{\frac{1}{2}}/(b^2-1) \tag{69}$$

and

$$R^{(e)}_1 > R^{(i)}_1 > R^{(o)}_1.$$

In this range, overstability first sets in with $b_0^2 = \frac{3}{2}$ and

$$R^{(o)}_{1,\,\mathrm{min.}} = 13{\cdot}5 \quad (\tfrac{27}{8} < T_1 \ll 27/8p^2). \tag{70}$$

For lower values of $T_1 > 1$, there exist values of b^2 for which overstability can occur and the minimum value of $R^{(o)}_1$ corresponds to the highest permissible $b_o^6 = T_1$. Thus,

$$R^{(o)}_{1,\,\mathrm{min.}} = 2T_1/(T_1^{\frac{1}{3}}-1) \quad (1 < T_1 < \tfrac{27}{8}). \tag{71}$$

Now (69) shows that $R^{(i)}_1$ is a minimum when $b^2 = 3$. However this cannot correspond to an unstable mode unless $T_1 > 27$; thus in the range $1 < T_1 < 27$, the minimum value of $R^{(i)}_1$ that *does* correspond to an unstable mode must have $b^6 = T_1$. Therefore

$$R^{(i)}_{1,\,\mathrm{min.}} = 2T_1/(T_1^{\frac{1}{3}}-1) \quad (1 < T_1 < 27), \tag{72}$$

while

$$R^{(i)}_{1,\,\mathrm{min.}} = 3(3T_1)^{\frac{1}{2}} \quad (27 < T_1 \ll 27/p^2). \tag{73}$$

When $T_1 = 27$, $R^{(i)}_{1,\,\mathrm{min.}} = T_1 = 2R^{(o)}_{1,\,\mathrm{min.}}$ and as T_1 increases thereafter both $R^{(e)}_{1,\,\mathrm{min.}}$ and $R^{(i)}_{1,\,\mathrm{min.}}$ become very much greater than $R^{(o)}_{1,\,\mathrm{min.}}$. In fact, $R^{(e)}_{1,\,\mathrm{min.}} = R^{(o)}_{1,\,\mathrm{min.}}$ when

$$T_1 = 5{\cdot}596 > \tfrac{27}{8}$$

and overstability sets in before the exchange of stabilities for all T_1 greater than 5·596. Now, when $b_e^2 = 3$, $R^{(e)}_{1,\,\mathrm{min.}} = T_1 = 27$: this is just the point where (73) first applies. We have already seen that neither overstability nor the transition to steady instability can occur for $T_1 < 1$; for $1 < T_1 < 27$, the exchange of stabilities occurs before the transition to steady instability, while for $27 < T_1$, this transition takes place first.

The onset of instability when $p \ll 1$ and $p^2 T_1 \ll 1$ is illustrated in figure 3 from the data assembled in table 3.

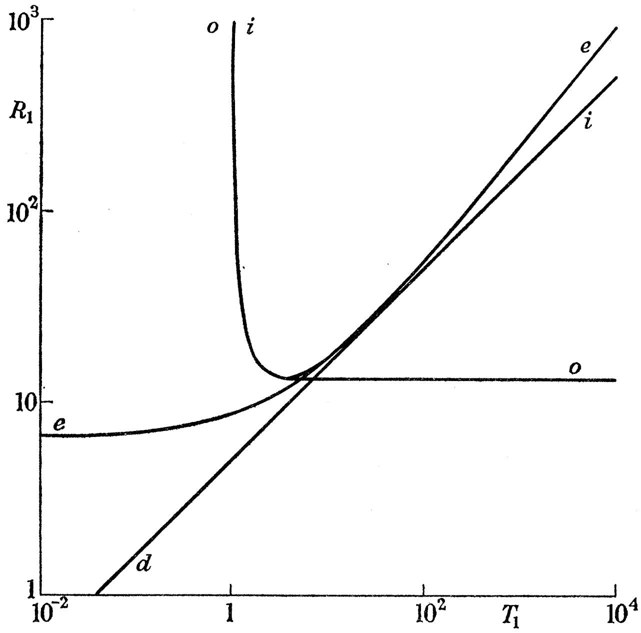

FIGURE 3. The onset of instability when $T_3 = p^2 T_1 \ll 1$. Curve *o* gives $R^{(o)}_{1,\text{min.}}$; *i* gives $R^{(i)}_{1,\text{min.}}$; *e* gives $R^{(e)}_{1,\text{min.}}$ and *d* gives $R^{(d)}_{,\text{min.}}$.

TABLE 3. ONSET OF INSTABILITY WHEN $T_3 = p_2 T_1 \ll 1$

T_1	$R^{(e)}_{1,\text{min.}}$	$R^{(i)}_{1,\text{min.}}$	$R^{(o)}_{1,\text{min.}}$
0	6·75	—	—
1	8·45	—	—
10	$1{\cdot}71 \times 10$	—	—
10^2	$5{\cdot}42 \times 10$	$5{\cdot}20 \times 10$	$1{\cdot}35 \times 10$
10^3	$2{\cdot}15 \times 10^2$	$1{\cdot}64 \times 10^2$	$1{\cdot}35 \times 10$
10^6	$1{\cdot}91 \times 10^4$	$5{\cdot}20 \times 10^3$	$1{\cdot}35 \times 10$
10^9	$1{\cdot}89 \times 10^6$	$1{\cdot}64 \times 10^5$	$1{\cdot}35 \times 10$
10^{12}	$1{\cdot}89 \times 10^8$	$5{\cdot}20 \times 10^6$	$1{\cdot}35 \times 10$

(ii) At the other extreme, when T_1 is very large, so that

$$(p^2 T_1)^{\frac{1}{3}} \gg 1, \tag{74}$$

$$R^{(e)}_{1,\text{min.}} = 3(\tfrac{1}{2}T_1)^{\frac{2}{3}} \quad \text{with} \quad b_e^6 = \tfrac{1}{2}T_1 \tag{75}$$

while

$$R^{(i)}_{1,\text{min.}} = 1{\cdot}5[2p(1-p)^2]^{\frac{1}{3}} R^{(e)}_{1,\text{min.}} \tag{76}$$

and

$$R^{(o)}_{1,\text{min.}} = 2[p^4/(1+p)]^{\frac{1}{3}} R^{(e)}_{1,\text{min.}}. \tag{77}$$

For a given b^2, overstability is possible only when $T_1 > (1+p)\, b^6/(1-p)$, so that (77) is only true for $p^2 < \frac{2}{3}$ or $p < 0{\cdot}81650$. If $p^2 > \frac{2}{3}$, overstability first appears at $R = R'$ where

$$R'_1 = \tfrac{2}{3}[\tfrac{1}{4}(1+p)^2\,(1-p)]^{-\frac{1}{3}} R^{(e)}_{1,\text{min.}}. \tag{78}$$

For $b^2 = b_i^2$, overstability can occur for some R_1 provided that $p < \frac{1}{3}$, and (76) will then apply. When $p = \frac{1}{3}$,

$$b_i^6 = b_e^6 = \tfrac{1}{2}T_1 \quad \text{and} \quad R^{(i)}_{1,\text{min.}} = R^{(e)}_{1,\text{min.}} \tag{79}$$

and for $p > \frac{1}{3}$, $R^{(i)}_{1,\text{min.}} = R'_1 > R^{(e)}_{1,\text{min.}}$. This behaviour of $R^{(i)}_{1,\text{min.}}$ and $R^{(o)}_{1,\text{min.}}$ for large T is shown in figure 4.

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(iii) The only common liquid that has $p \ll \frac{1}{3}$ and so exhibits the behaviour described above is mercury, for which $p = 0{\cdot}025$. In order to illustrate the transitions between different modes of instability the corresponding values of R_1 have been calculated.

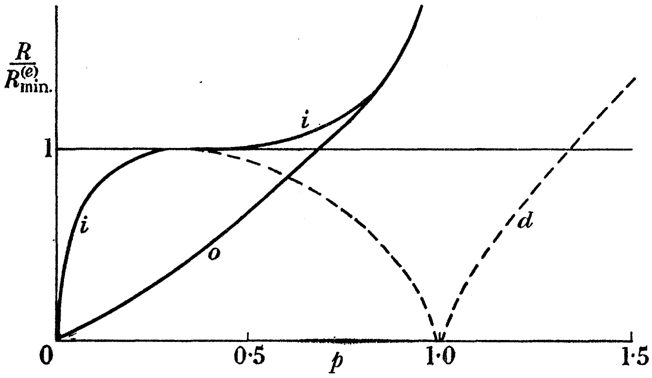

FIGURE 4. The onset of instability as the Prandtl number is varied, for large T_1. Curve *o* has $R = R^{(o)}_{\text{min.}}$; *i* has $R = R^{(i)}_{\text{min.}}$ and *d* has $R = R^{(d)}_{\text{min.}}$.

The behaviour for $b^2 = 3$ has already been illustrated in figure 1; the values of R_1 for the onset of the various modes of instability as T_1 varies are displayed in figure 5 and table 4 on p. 112.

(*d*) *The case when* $p \ll 1$

Most of the preceding results are valid for all values of the Prandtl number that permit the relevant motions; in astrophysical applications the effective Prandtl number turns out to be very much less than unity owing to the efficiency of radiative transfer at such high temperatures. If this is so, then the Rayleigh number

$$R_1 = g\alpha\beta d^4/\pi^4\kappa\nu$$

and the Taylor number

$$T_1 = 4\Omega^2 d^4/\pi^4\nu^2$$

will both become extremely large and it is therefore convenient to replace them by the modified Rayleigh and Taylor numbers

$$R_3 = pR_1 = g\alpha\beta d^4/\pi^4\kappa^2 \tag{80}$$

and

$$T_3 = p^2 T_1 = 4\Omega^2 d^4/\pi^4\kappa^2 \tag{81}$$

neither of which involves the viscosity ν.

We can express the results already derived in terms of these modified numbers. Then

$$R_3^{(e)} = \frac{1}{p(b^2-1)}\,[p^2 b^6 + T_3] \tag{82}$$

and

$$R_3^{(o)} = \frac{2p(1+p)}{(b^2-1)}\left[b^6 + \frac{T_3}{(1+p)^2}\right], \tag{83}$$

while for

$$T_3 \gg (1-p)^2,$$

$$R^{(i)}_{3,\text{min.}} = \tfrac{9}{4}[2(1-p)\,T_3]^{\frac{2}{3}} \tag{84}$$

and

$$(1-p)^2\, b_i^6 = 2T_3. \tag{85}$$

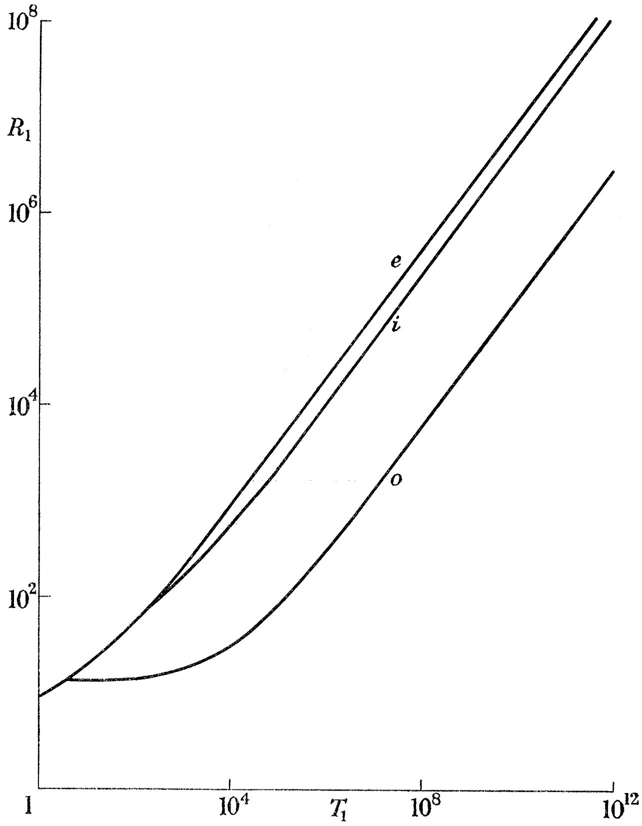

FIGURE 5. The onset of instability in mercury ($p = 0{\cdot}025$). Curve o gives $R^{(o)}_{1,\min.}$; i gives $R^{(i)}_{1,\min.}$ and e gives $R^{(e)}_{1,\min.}$.

TABLE 4. ONSET OF INSTABILITY IN MERCURY ($p = 0{\cdot}025$)

T_1	$R^{(e)}_{1,\min.}$	$R^{(i)}_{1,\min.}$	$R^{(o)}_{1,\min.}$
10	$1{\cdot}71 \times 10$	—	—
10^2	$5{\cdot}42 \times 10$	$5{\cdot}20 \times 10$	$1{\cdot}35 \times 10$
10^3	$2{\cdot}15 \times 10^2$	$1{\cdot}64 \times 10^2$	$1{\cdot}65 \times 10$
10^4	$9{\cdot}29 \times 10^2$	$5{\cdot}34 \times 10^2$	$3{\cdot}00 \times 10$
10^6	$1{\cdot}91 \times 10^4$	$1{\cdot}06 \times 10^4$	$3{\cdot}17 \times 10^2$
10^8	$4{\cdot}08 \times 10^5$	$2{\cdot}26 \times 10^5$	$5{\cdot}47 \times 10^3$
10^{10}	$8{\cdot}77 \times 10^6$	$4{\cdot}83 \times 10^6$	$1{\cdot}29 \times 10^5$
10^{12}	$1{\cdot}89 \times 10^8$	$1{\cdot}05 \times 10^8$	$2{\cdot}76 \times 10^6$

If the viscosity is negligible compared with the thermal conductivity, so that

$$p \ll 1 \tag{86}$$

(83) reduces to

$$R^{(o)}_3 = 2p[b^6 + T_3]/(b^2 - 1). \tag{87}$$

Then for $T_3 \ll 1$,

$$R^{(o)}_{3,\min.} = \tfrac{27}{2}p \quad \text{with } b_o^2 = \tfrac{3}{2}, \tag{88}$$

while as $T_3 \to \infty$ we have

$$R^{(o)}_{3,\min.} = 6p(\tfrac{1}{2}T_3)^{\frac{2}{3}} \quad \text{with} \quad b_o^6 = \tfrac{1}{2}T_3.\text{*} \tag{89}$$

Also, if $T_3 \ll b^6$

$$R^{(i)}_3 = 2b^3 T_3^{\frac{1}{2}}/(b^2 - 1) \tag{90}$$

* Chandrasekhar (1961, p. 129) discusses the case when $p \ll 1$. However, he has not considered the point that T_1 becomes infinite if we allow ν to go to zero, so that his discussion is valid only for $T_3 \ll 1$.

and so $$R^{(i)}_{3,\,\text{min.}} = 3(3T_3)^{\frac{1}{2}} \quad \text{with} \quad b_i^2 = 3, \tag{91}$$

while if $T_3 \gg 10^3$ $$R^{(i)}_{3,\,\text{min.}} = \tfrac{9}{4}(2T_3)^{\frac{2}{3}} \quad \text{and} \quad b_i^6 = 2T_3. \tag{92}$$

We see from (82) that the exchange of stabilities takes place at $R_3 = \frac{27}{4}p \ll 1$ if $T_3 = 0$ but that, for a finite value of T_3, $R_3^{(e)}$ increases as (T_3/p) and rapidly becomes infinite. On the other hand, from (88), overstability will set in at very small R_3 unless $T_3 \gg p^{-\frac{3}{2}}$, while instability follows at finite R_3 given by (90) or (92). We can simplify these results by supposing that the viscosity is identically zero, though it then becomes necessary to replace

$$s = d^2\sigma/\pi^2\nu$$

by $$s_1 = ps = d^2\sigma/\pi^2\kappa. \tag{93}$$

We then obtain a characteristic equation

$$s_1^3 + As_1^2 + Bs_1 + C = 0, \tag{94}$$

where $$A = b^2, \tag{95}$$

$$B = (1/b^2)\,[T_3 - R_3(b^2-1)] \tag{96}$$

and $$C = T_3. \tag{97}$$

Then $C \geqslant 0$ always and $AB < C$ if $R_3 > 0$ so that unstable solutions exist throughout the positive quadrant of the R_3–T_3 plane, for all values of b^2. In particular, the exchange of stabilities will only take place at $T_3 = 0$—that is, if there is no rotation—when there is a state of marginal stability. (This degenerate state persists for all R_3 if $T_3 = 0$; however, of the other two solutions, one is always unstable for $R_3 > 0$.) On the other hand, overstability is possible for all values of T_3. Thus the criterion for the discriminant to be zero will everywhere be the condition for the onset of steady instability: this is given by (90) to (92).

Physically, one is interested in the lowest temperature gradient that will produce convection: for an inviscid fluid with finite thermal conductivity we see that, while there is a state of marginal stability which develops into an unstable solution with an infinitesimal temperature gradient if there is no rotation, for even a small Taylor number this marginal state itself becomes impossible. Equation (82) shows that viscosity plays a dual role: in a non-rotating system it makes marginal stability more difficult to obtain; but if rotation is imposed, it is only through the influence of viscous effects that such a state can occur. The exchange of stabilities is then an essentially viscous phenomenon.

On the other hand, overstability and the ensuing instability when $\Delta \leqslant 0$ are phenomena characteristic of an inviscid fluid: as the Prandtl number increases, their behaviour is modified until they are finally extinguished. When $p = 0$, overstability occurs for an infinitesimal temperature gradient, while instability will follow when the temperature gradient has grown sufficiently for R_3 to satisfy (91) or (92); these conditions depend on the thermometric conductivity only and are not affected by the presence of a viscosity small compared with this.

(e) *The time-dependence for overstability and instability*

When $R_1 = 0$ the characteristic equation gives one steady and two oscillatory modes, all damped $$s = -b^2/p \quad \text{or} \quad s = -b^2 \pm \mathrm{i}(T_1/b^2)^{\frac{1}{2}}. \tag{98}$$

At the other extreme, when $R_3 \gg T_3, b^2$, the solutions are

$$s = -b^2 \quad \text{or} \quad s = \pm\left[\frac{b^2-1}{pb^2}R_1\right]^{\frac{1}{2}}. \tag{99}$$

The intermediate behaviour of the solutions has already been depicted in figure 2.

The frequency of overstable oscillations and the growth rate of instabilities are given by the time constant σ. When $p \ll 1$, we may use (94) to (97) to calculate the value of σ at the onset of overstability and of steady instability.

(i) *Overstability*

For overstability, if $p^2 \ll T_3 \ll \frac{27}{8}$,

$$s_1 = \pm \mathrm{i}(\tfrac{2}{3}T_3)^{\frac{1}{2}} \quad \text{or} \quad \sigma = \pm \mathrm{i}(\tfrac{2}{3})^{\frac{1}{2}}\, 2\Omega. \tag{100}$$

(This agrees with the result given by Chandrasekhar (1961, p. 129) except that this derivation is only valid when $T_3 \ll 1$.)

For $T_3 \gg 1$,

$$s_1 = \pm \mathrm{i}\surd 2(\tfrac{1}{2}T_3)^{\frac{1}{3}} \quad \text{or} \quad \sigma = \pm \mathrm{i}\surd 2\pi^2\kappa d^{-2}(\tfrac{1}{2}T_3)^{\frac{1}{3}} \tag{101}$$

or, alternatively, if we introduce the characteristic time for relaxation under thermal conductivity

$$\tau_\kappa = d^2/\pi^2 b^2 \kappa \tag{102}$$

then

$$\sigma = \pm \mathrm{i}\surd 2/\tau_\kappa. \tag{103}$$

(ii) *Instability*

For instability, if $p^2 \ll T_3 \ll 1$,

$$s_1 = (\tfrac{1}{3}T_3)^{\frac{1}{2}} \quad \text{or} \quad \sigma = 2\Omega/\surd 3 \tag{104}$$

while, if $T_3 \gg 1$,

$$s_1 = \tfrac{1}{2}(2T_3)^{\frac{1}{3}} \quad \text{or} \quad \sigma = \tfrac{1}{2}\pi^2\kappa d^{-2}(2T_3)^{\frac{1}{3}} \tag{105}$$

or, alternatively,

$$\sigma = 1/2\tau_\kappa. \tag{106}$$

Hence we see that for slow rotations ($T_3 \ll 1$) the period of the oscillations in the marginal state and the time-constant of the first mode of instability are both very near to the period of rotation ($2\pi/\Omega$) and independent of κ; while for rapidly rotating systems the period of the oscillations and the growth-rate of the first mode of steady instability correspond to the thermal relaxation time τ_κ.

(iii) *Time constants beyond the initial onset of instability*

We shall only consider the case when $T_3 \gg 1$. The modes with $b^6 \ll \frac{1}{2}T_3$ first become overstable with

$$s_1 = \pm \mathrm{i}(T_3^{\frac{1}{2}}/b) \quad \text{or} \quad \sigma = \pm \mathrm{i}(2\Omega/b). \tag{107}$$

Similarly, instability sets in for these graver modes with

$$s_1 = (\tfrac{1}{2}T_3)^{\frac{1}{3}} \quad \text{or} \quad \sigma = \pi^2\kappa d^{-2}(\tfrac{1}{2}T_3)^{\frac{1}{3}}. \tag{108}$$

Thus the gravest modes of overstability have a frequency of the same order as Ω and inversely proportional to b, while they develop into steady instabilities with a growth-rate independent of b but proportional to the cube root of κ.

(f) *The nature of the horizontal motions*

For an incompressible fluid we have

$$\operatorname{div} \mathbf{u} = 0 \tag{109}$$

and the effects of rotation are best demonstrated by taking the curl of the equation of motion, whose vertical component gives

$$\frac{\partial \zeta}{\partial t} = 2\Omega \frac{\partial w}{\partial z} + \nu \nabla^2 \zeta, \tag{110}$$

where $\mathbf{i}_z$ is a unit vector along the z axis and

$$\mathbf{u}.\mathbf{i}_z = w, \quad \operatorname{curl} \mathbf{u}.\mathbf{i}_z = \zeta. \tag{111}$$

The form of the normal modes at the exchange of stabilities has already been obtained (Veronis 1959; Chandrasekhar 1961). However, if the Prandtl number is less than one-third, instability sets in earlier and so Veronis's results cannot be applied. We shall here extend his treatment to cover the onset of overstability and of non-oscillatory instability.

Let us, then, assume free boundaries and take a disturbance of the form

$$w = W \cos k_x x \cos k_y y \sin k_z z \exp[(\lambda + \mathrm{i}\mu) t], \tag{112}$$

where
$$k_z = \frac{\pi}{d}, \quad k_x = \frac{\pi}{a_x} \quad \text{and} \quad k_y = \frac{\pi}{a_y}. \tag{113}$$

Then
$$k^2 = k_x^2 + k_y^2 \quad \text{and} \quad b^2 = 1 + d^2k^2/\pi^2 \quad \text{as before.} \tag{114}$$

When we substitute into the linearized equations, we force the other perturbed quantities to show a similar variation with space and time, though they may be out of phase with w. It is convenient to separate the horizontal motions into two components, $\mathbf{u}_1$ and $\mathbf{u}_2$, where

$$\operatorname{div} \mathbf{u}_1 = -\partial w/\partial z, \quad \operatorname{div} \mathbf{u}_2 = 0,$$

$$\operatorname{curl} \mathbf{u}_1.\mathbf{i}_z = 0, \quad \operatorname{curl} \mathbf{u}_2.\mathbf{i}_z = \zeta.$$

Then
$$\mathbf{u}_1 = -(k_x \sin k_x x \cos k_y y \,\mathbf{i}_x + k_y \cos k_x x \sin k_y y \,\mathbf{i}_y) \frac{W \cos k_z z}{k_z(b^2-1)} \mathrm{e}^{(\lambda+\mathrm{i}\mu)t} \tag{115}$$

and remains in phase with w, while

$$\mathbf{u}_2 = -U_2 \cos k_z z (k_y \cos k_x x \sin k_y y \,\mathbf{i}_x - k_x \sin k_x x \cos k_y y \,\mathbf{i}_y), \tag{116}$$

where
$$U_2 = \frac{2\Omega W \exp[(\lambda + \mathrm{i}\mu) t - \mathrm{i}\Phi]}{k_z(b^2-1)\,[(\lambda+\tau_\nu)^2 + \mu^2]^{\frac{1}{2}}}, \tag{117}$$

$$\tau_\nu = \frac{d^2}{\pi^2 b^2 \nu} = \frac{1}{k_z^2 b^2 \nu} \tag{118}$$

and
$$\tan \Phi = \mu/(\lambda + \tau_\nu). \tag{119}$$

Therefore in a non-rotating system $\mathbf{u}_2 = 0$ and the effect of rotation is to introduce this additional motion (which has the form of horizontal whirls) without altering the simple convective circulation $\mathbf{u}_1$. Moreover, when oscillatory disturbances occur $\mathbf{u}_2$ will be out of phase with $\mathbf{u}_1$ and with $\mathbf{w}$.

Suppose for the moment that $\Omega = 0$: then the exchange of stabilities first occurs with $b^2 = \frac{3}{2}$ and if we assume square cells, so that $k_x = k_y = \frac{1}{2}k_z$, then

$$\mathbf{u}_1 = -W(\sin k_x x \cos k_y y\, \mathbf{i}_x + \cos k_x x \sin k_y y\, \mathbf{i}_y) \cos k_z z \tag{120}$$

and the horizontal flow has the same maximum speed W as the vertical motions. Conversely, if we make the plausible assumption that in convective motion the horizontal and vertical speeds have approximately the same amplitude at a given instant it is easy to demonstrate that the horizontal 'radius' or half-width a of the cell must be twice its depth. For $\operatorname{div} \mathbf{u} = 0$ gives $a = 2d$ whence $b^2 = \frac{3}{2}$ once more. Alternatively, if $|\mathbf{u}_1| \sim U_1$ and $|w| \sim W$,

$$\frac{2a}{U_1} \approx \frac{d}{W}$$

so that the typical times taken for a fluid element to cross the cell horizontally and to rise from the bottom to the top of the layer are equal. If $b^2 \gg 1$, so that the cell is attenuated, then the horizontal velocity will be inversely proportional to b and so much less than W; it should be noted that these motions persist even after Coriolis effects have added $\mathbf{u}_2$ (which may be far greater than $\mathbf{u}_1$) and that the time spent in traversing the cell horizontally remains of the same order as that devoted to vertical motion.

We must next consider the velocity field $\mathbf{u}_2$ induced by the rotation; its streamlines are given by

$$\mathrm{d}y/\mathrm{d}x = -\tan k_x x \cot k_y y$$

and the resulting flow pattern consists of slightly distorted circular eddies centred on the rising and falling currents (maxima of $|w|$ in the horizontal plane). Although these eddies do not promote heat transport, they will be far more prominent than the convective circulation in a rapidly rotating system, where the trajectory of a fluid element will be as given by Veronis (1959).

Fortunately, we can simplify (116) and (117) by assuming that $p \ll 1$ and restricting our attention to three important cases, namely

(i) $|\mu| \gg \lambda, \tau_v$ which corresponds to the onset of overstability when $p \ll 1$,

(ii) $\lambda \gg \tau_v, |\mu|$ which corresponds to the transition from overstable to non-oscillatory motions when $p \ll 1$, and

(iii) $\tau_v \gg \lambda, |\mu|$ which corresponds to the exchange of stabilities.

We shall now consider these cases separately.

(i) $\mu > 0$; $\lambda = \tau_v = 0$.

Then if we take

$$w = W \cos k_x x \cos k_y y \sin k_z z \cos \mu t \tag{121}$$

we have two values of μ, corresponding to the two complex conjugate modes, and

$$\Phi = \pm \tfrac{1}{2}\pi \tag{122}$$

whence, from equation (116), writing μ for $|\mu|$ we have

$$U_2 = 2\Omega W \sin \mu t / \mu k_z (b^2 - 1) \tag{123}$$

in either case. So we see that the eddy motions $\mathbf{u}_2$ oscillate in quadrature with the convective velocities w and $\mathbf{u}_1$.

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At the onset of overstability, μ is given by (100) or (101) Then, for $p^2 \ll T_3 \ll \frac{27}{8}$,

$$U_2 = \sqrt{6}\, W \sin \mu t / k_z$$

and for square cells $$\mathbf{u}_2 = -(\tfrac{3}{2})^{\frac{1}{2}}\, W \sin \mu t\, \mathbf{u}_0, \tag{124}$$

where $$\mathbf{u}_0 = \cos k_z z(\cos k_x x \sin k_y y\, \mathbf{i}_x - \sin k_x x \cos k_y y\, \mathbf{i}_y) \tag{125}$$

and the magnitude of $\mathbf{u}_2$ is almost the same as that of w. (Since $b_o^2 = \frac{3}{2}$ the same is true of $\mathbf{u}_1$ by virtue of the arguments leading to equation (120).)

Likewise, when $T_3 \gg \frac{27}{8}$ $$U_2 = W \sin \mu t / k_z (\tfrac{1}{2} T_3)^{\frac{1}{6}} \tag{126}$$

and for square cells $$\mathbf{u}_2 = -(\tfrac{1}{2})^{\frac{1}{2}}\, W \sin \mu t\, \mathbf{u}_0 \tag{127}$$

which has the same magnitude as w. In this case, though, the amplitude of $\mathbf{u}_1$ is very much less.

When $T_3 \gg 1$ and $b^6 \ll \frac{1}{2} T_3$ overstability sets in for a given b^2 with $\mu = 2\Omega/b$, according to equation (107), and then $$U_2 = bW \sin \mu t / (b^2 - 1)\, k_z. \tag{128}$$

For a square cell $$\mathbf{u}_2 = -\{b/[2(b^2-1)]^{\frac{1}{2}}\}\, W \sin \mu t\, \mathbf{u}_0 \tag{129}$$

so that the magnitude of $\mathbf{u}_2$ is once again nearly equal to W.

For any overstable model beyond its first appearance, λ will be positive and the phase shift between $\mathbf{u}_1$ and $\mathbf{u}_2$ will diminish until it becomes zero as the oscillations give way to steadily increasing motions, described in the next paragraph.

(ii) $\lambda > 0$; $\tau_\nu = \mu = 0$.

Let us take $$w = W \cos k_x x \cos k_y y \sin k_z z\, \mathrm{e}^{\lambda t} \tag{130}$$

so that $$U_2 = 2\Omega W \mathrm{e}^{\lambda t} / \lambda k_z (b^2 - 1). \tag{131}$$

At the onset of steady instability, λ is given by (104) or (105). Then, for $p^2 \ll T_3 \ll 1$,

$$U_2 = \sqrt{3}\, W \mathrm{e}^{\lambda t} / 2k_z \tag{132}$$

and for square cells $$\mathbf{u}_2 = -\tfrac{1}{2}\sqrt{3}\, \mathbf{u}_0\, \mathrm{e}^{\lambda t}. \tag{133}$$

Again, for $T_3 \gg 1$, $$U_2 = \sqrt{2}\, W \mathrm{e}^{\lambda t} / k_z (2T_3)^{\frac{1}{6}} \tag{134}$$

whence, for square cells, $$\mathbf{u}_2 = W \mathbf{u}_0\, \mathrm{e}^{\lambda t}. \tag{135}$$

So, once more, the eddy motions have the same magnitude as the vertical motions at the onset of instability.

As before, we can see what happens when $T_3 \gg 1$ and $b^6 \ll 2T_3$: we now find that instability sets in for a given b^2 with

$$U_2 = \sqrt{2}\, (\tfrac{1}{2} T_3)^{\frac{1}{6}}\, W \mathrm{e}^{\lambda t} / k_z (b^2 - 1). \tag{136}$$

However, we now find for our square cell that

$$\mathbf{u}_2 = (\tfrac{1}{2} T_3)^{\frac{1}{6}}\, W \mathbf{u}_0\, \mathrm{e}^{\lambda t} / (b^2 - 1)^{\frac{1}{2}} \tag{137}$$

and the amplitude of the eddy motions becomes much greater than W in this linear approximation.

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(iii) $\tau_\nu > 0$; $\lambda = \mu = 0$.

This case is the exchange of stabilities; the behaviour of $\mathbf{u}_1$ and $\mathbf{u}_2$ (derived by Veronis using different terminology) is repeated here for the sake of completeness. With

$$w = W \cos k_x x \cos k_y y \sin k_z z,$$

$$U_2 = \frac{2\Omega W}{\tau_\nu k_z(b^2-1)} = \frac{T_1^{\frac{1}{2}} W}{b^2(b^2-1)\,k_z} \tag{138}$$

and for $T_1 \gg 1$ the first marginal state has

$$U_2 = \surd 2\, W/k_z(\tfrac{1}{2}T_1)^{\frac{1}{6}}; \tag{139}$$

then, for a square cell,

$$\mathbf{u}_2 = W\mathbf{u}_0. \tag{140}$$

However, for a state of marginal stability with $b^6 \ll \frac{1}{2}T_1$

$$\mathbf{u}_2 = \left[\frac{T_1}{2b^4(b^2-1)}\right]^{\frac{1}{2}} W\mathbf{u}_0 \tag{141}$$

which has a much greater amplitude than w.

From the preceding discussion we see that at the initial onset of instability the horizontal and vertical velocities have more or less the same amplitude. This holds whether overstability, steady instability or the exchange of stabilities is being considered. Beyond this initial onset, graver modes become unstable in which the eddies have a much greater velocity than the convection driving them (except in the case of overstability). This seems incongruous and one might suspect that non-linear theory should modify this result—we shall return to it in part II.

We have already observed that the time taken for an element to cross the cell is more or less the same as the time taken for it to rise through the layer; if its speed is approximately the same in either case—which holds at the onset of instabilities—then the trajectory of the fluid element must have about the same length in a horizonal as in a vertical plane. We can express this point, alternatively, by saying that although rotation distorts the trajectory (and with it the boundary of the convection cell), it does not appreciably alter the length measured along a streamline (or the length of the contorted boundary of the cell). This fact was noticed by Veronis (1959).

(g) *Conclusion*

In this section the onset of convection in a rotating fluid has been studied in some detail, using the general methods outlined in § 2. This has provided a new approach to the onset of instability, based on the characteristic equation and enabled us to derive Chandrasekhar's results for overstability and the exchange of stabilities very simply.

The most important parts of this section are those dealing with the transition from oscillatory to steady motions, which has not hitherto been discussed for a rotating system. In § 3 (b) we found the value $R^{(d)}$ of the Rayleigh number above which oscillatory perturbations were not solutions of the linearized equations. Next, we distinguished the range over which this corresponded to a transition from overstable to non-oscillatory instabilities and showed that there is a range over which steady instability sets in *before* the exchange of stabilities, provided that $p < \frac{1}{3}$; in this range

$$R^{(o)}_{\text{min.}} < R^{(i)}_{\text{min.}} < R^{(e)}_{\text{min.}}.$$

These discussions have most application when $p \ll 1$ as in astrophysical problems, and in that case a simplified treatment can be given. As the viscosity ν tends to zero, the criterion for the onset of instability becomes independent of ν and can be expressed as a relation between R_3 and T_3 only, both of which depend solely on the thermometric conductivity κ.

Under these conditions the time constants for steady and oscillatory motions can be found. When $T_3 \ll 1$, the motions are governed by the period of rotation of the system but when $T_3 \gg 1$ their characteristic time is proportional to the thermal decay time for instability. From these time constants it is possible to find the ratio of the vortical to the circulatory speeds that arise in a rotating system. At the onset of instability these speeds are comparable and they remain so for all modes of overstability. However, the linearized equations imply that the eddy velocities are much greater than convective velocities for steady instabilities.

The importance of distinguishing the transition from overstable to steady motions arises from the study of heat transport. Experiments by Goroff (1960), and observations by Danielson (1961 *a*, *b*) of the magnetic analogue, show that overstable oscillations carry only a negligible amount of heat. Goroff's results have been interpreted as showing that little convection occurs until an exchange of stabilities is possible. On the basis of arguments given here, the criterion should be that $R^{(i)}_{\text{min.}}$ rather than $R^{(e)}_{\text{min.}}$ is reached; the gravest modes become unstable with $R \propto T$ and the ratio of $R^{(i)}$ to $R^{(e)}$ is then equal to the Prandtl number: for mercury this is only a factor of 40 but in the sun it becomes one of 10^{10}; such a quantity is scarcely to be sneezed at.

4. The onset of convection in the presence of a magnetic field

(*a*) *The onset of overstability and the exchange of stabilities*

The methods of § 2 are used here to derive the relevant criteria for convection in a magnetic field; the treatment will thus resemble that of the previous section—except that it is shorter.

The onset of convection in a magnetic field was first considered by Thompson (1951) whose analysis was later elaborated by Chandrasekhar (1952, 1954). The onset of overstability and the exchange of stabilities are treated at length in Chandrasekhar (1961). Danielson (1961 *b*) first discussed the transition to non-oscillatory motion in connexion with the penumbral structure of sunspots. He obtained the characteristic equation—which is also implicit in Chandrasekhar's work—and then searched for $R^{(i)}$ under certain approximations. He assumed

$$p_1 \ll p_2 \ll 1 \tag{142}$$

and then made further assumptions which enabled him to find values for $R^{(i)}$ when the Chandrasekhar number Q is very small or very large. We shall here obtain a general condition for the discriminant to be negative, valid for all p_1 and p_2, before proceeding to the astrophysical situation where (142) can be assumed to hold. Then we investigate the behaviour of $R^{(i)}$ at extreme values of Q, when $Q \ll Q_0$ or $Q \gg Q_0$, where

$$Q_0 = p_2 b^4/p_1^2. \tag{143}$$

In the former range our results agree with those of Danielson; in the latter there is a trivial discrepancy. Danielson's paper is also concerned with the linear growth rate, which he treats as a time constant for the system. This demands an understanding of the non-linear problem and will be discussed elsewhere (Weiss 1964).

The characteristic equation may be derived as before (Chandrasekhar 1961); we suppose a vertical magnetic field **H** in a fluid with conductivity σ, introduce the resistivity

$$\eta = (4\pi\mu\sigma)^{-1} \tag{144}$$

and allow a perturbation **u**, **h**, ϑ in the velocity, magnetic field and temperature. Then, defining the modified Chandrasekhar number

$$Q_1 = \frac{Q}{\pi^2} = \frac{\mu H^2 d^2}{4\pi^3 \rho\eta\nu} \tag{145}$$

we have

$$s^3 + As^2 + Bs + C = 0 \tag{146}$$

with

$$A = (1/p_1 p_2)(p_1 + p_2 + p_1 p_2)\, b^2, \tag{147}$$

$$B = (1/p_1 p_2)[(1 + p_1 + p_2)\, b^4 + p_1 Q_1 - p_2 R_2], \tag{148}$$

$$C = (b^2/p_1 p_2)[b^4 + Q_1 - R_2], \tag{149}$$

and $R_2 = (b^2 - 1)\, R_1/b^2$ as before. Then, from (11) and (12),

$$R_1^{(e)} = b^2(b^4 + Q_1)/(b^2 - 1) \tag{150}$$

and

$$R_1^{(o)} = \frac{(1+p_2)(p_1+p_2)}{p_2^2} \frac{b^2}{b^2-1}\left[b^4 + \frac{p_1^2}{(1+p_1)(p_1+p_2)} Q_1\right] \tag{151}$$

as stated by Chandrasekhar. Once again, it is possible to derive explicit relationships between Q and $R^{(e)}_{\text{min.}}$ or $R^{(o)}_{\text{min.}}$:

$$Q_1 = R^{(e)}_{1,\text{min.}} - 3(\tfrac{1}{2}R^{(e)}_{1,\text{min.}})^{\frac{2}{3}} \tag{152}$$

and

$$Q_1 = \frac{(1+p_1)(p_1+p_2)}{p_1^2}\left[\frac{p_2^2 R^{(o)}_{1,\text{min.}}}{(1+p_2)(p_1+p_2)} - 3\left\{\frac{p_2^2 R^{(o)}_{1,\text{min.}}}{2(1+p_2)(p_1+p_2)}\right\}^{\frac{2}{3}}\right] \tag{153}$$

which reduce to

$$R^{(e)}_{1,\text{min.}} \sim Q_1 \quad \text{and} \quad R^{(o)}_{1,\text{min.}} = \frac{(1+p_2)\, p_1^2}{(1+p_1)\, p_2^2} Q_1 \tag{154}$$

as $Q_1 \to \infty$.

Provided that $p_2 > p_1$ (i.e. $\kappa > \eta$) overstability is possible and first occurs for a given b^2 with

$$Q_1 = \frac{1+p_1}{p_2 - p_1} b^4 \tag{155}$$

at

$$R_1 = \frac{1+p_2}{p_2-p_1} \frac{b^6}{b^2-1}. \tag{156}$$

If $p_2 > p_1$ there will exist some range of Q_1 over which overstability sets in at a lower Rayleigh number than the exchange of stabilities.

(b) The transition from oscillatory to steady motions

From (146) to (149) we find that the quantities defined in § 2 are given by

$$E = (1/3p_1^2p_2^2)\,[\{-(p_1^2+p_2^2)+p_1p_2(1+p_1+p_2-p_1p_2)\}\,b^4+3p_1p_2(p_1Q_1-p_2R_2)] \quad (157)$$

and

$$F = (b^2/27p_1^3p_2^3)\,[\{2(p_1^3+p_2^3+p_1^3p_2^3)-3p_1p_2(p_1^2+p_2^2+p_1+p_2+p_1^2p_2+p_1p_2^2-4p_1p_2)\}\,b^4 \\ +9p_1p_2\{(2p_2-p_1-p_1p_2)\,p_1Q_1+(p_2-2p_1+p_1p_2)\,p_2R_2\}], \quad (158)$$

whence we can evaluate the discriminant

$$\Delta = 4E^3+27F^2.$$

After a not inconsiderable amount of algebra, we obtain

$$\Delta = 27p_1^2p_2^2[(p_1-p_2)\,(Kb^{12}+Lb^8)+Mb^4+N], \quad (159)$$

where

$$K = (p_2-p_1)\,[1-(p_1+p_2)+p_1p_2]^2, \quad (160)$$

$$L = 2[\{(p_1^2+2p_1p_2-2p_2^2)-p_1(p_1^2+6p_1p_2-4p_2^2)+p_1^2p_2(4p_1-p_2)-p_1^3p_2^2\}\,Q_1 \\ -\{(2p_1^2-2p_1p_2-p_2^2)-p_2(4p_1^2-6p_1p_2-p_2^2)+p_1p_2^2(p_1-4p_2)+p_1^2p_2^3\}\,R_2], \quad (161)$$

$$M = \{(-p_1^2+8p_2^2-8p_1p_2)+p_1p_2(10p_1-8p_2)-p_1^2p_2^2\}\,p_1^2Q_1^2 \\ +2\{(10p_1^2-19p_1p_2+10p_2^2)-p_1p_2(p_1+p_2)+p_1^2p_2^2\}\,p_1p_2Q_1R_2 \\ +\{(8p_1^2-8p_1p_2-p_2^2)-p_1p_2(8p_1-10p_2)-p_1^2p_2^2\}\,p_2^2R_2^2 \quad (162)$$

and $N = 4p_1p_2(p_1Q_1-p_2R_2)^3.$ (163)

The condition for the roots to be real is that

$$\Delta \leqslant 0.$$

Since the complete expression for Δ is rather unwieldy, let us immediately adopt some simplifying assumptions. The condition $p_2 > p_1$ is not satisfied by laboratory materials and this theory can only find astrophysical applications. Under the conditions prevalent in the sun and other stars, $p_1 \ll p_2 \ll 1$; we shall therefore assume this in order to simplify the criterion (161). (Since the leading terms in K and L cancelled out, it would not have been possible to assume (142) *before* deriving (159) to (163).) Then if we write

$$Q_2 = p_1Q_1, \quad R_4 = p_2R_2, \quad p_3 = p_2/p_1 = \kappa/\eta \gg 1 \quad (164)$$

the condition for the existence of real roots becomes

$$f \leqslant 0, \quad (165)$$

where

$$f = -[\{b^4-2(2p_3Q_2-R_4)\}\,b^4-\{8Q_2^2+20Q_2R_4-R_4^2\}]\,b^4+4(Q_2-R_4)^3/p_3. \quad (166)$$

As in § 3 there is no readily attainable general solution, so we shall consider the extreme cases when the Chandrasekhar number is either very small or very large.

(i) Suppose that $Q_1 \ll Q_0$, i.e. $Q_2 \ll p_3b^4$. (167)

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It is convenient to alter the notation again by introducing

$$R_5 = p_2 R_1 = p_3 R_3 \tag{168}$$

and

$$Q_4 = p_2 Q_1 = p_3 Q_2 = q^2. \tag{169}$$

Then it can be shown that oscillatory solutions exist for all

$$R_5 < R_5^{(d)} = b^4[2Q_4^{\frac{1}{2}} - b^2]/(b^2-1). \tag{170}$$

So no oscillations are possible at all unless

$$Q_4 > \tfrac{1}{4}b^4$$

and it can be demonstrated that

$$R^{(i)}_{5,\,\mathrm{min.}} = q^3/(q-1) \quad (1 < q < \tfrac{9}{2}), \tag{171}$$

while

$$R^{(i)}_{5,\,\mathrm{min.}} = \frac{b_i^4}{b_i^2-1}(2q-b_i^2) \quad (\tfrac{9}{2} < q \ll p_3), \tag{172}$$

where

$$b_i^2 = \tfrac{1}{4}[(2q+3) - \{(2q-9)(2q-1)\}^{\frac{1}{2}}]. \tag{173}$$

For $1 \ll q \ll p_3$, (172) and (173) reduce to

$$R^{(i)}_{5,\,\mathrm{min.}} = 8Q_4^{\frac{1}{2}}, \quad b_i^2 = 2. \tag{174}$$

Now the exchange of stabilities first occurs at

$$R^{(e)}_{5,\,\mathrm{min.}} = Q_4 \quad (Q_3 > 1) \tag{175}$$

and so $R^{(i)}_{\mathrm{min.}} = R^{(e)}_{\mathrm{min.}}$ when $q = \frac{27}{4}$ and then $R_5 = Q_4 = 45{\cdot}56$, while $b_i^2 = \frac{9}{4}$. Thus for $Q_4 < \frac{729}{16}$, $R^{(e)}_{\mathrm{min.}} < R^{(i)}_{\mathrm{min.}}$ and steady convection sets in at the exchange of stabilities curve; but for $Q_4 > \frac{729}{16}$, $R^{(e)}_{\mathrm{min.}} > R^{(i)}_{\mathrm{min.}}$ and steady convection sets in at the instability curve first.

(ii) Now consider the opposite extreme and suppose that the Chandrasekhar number is sufficiently large for

$$Q_1 \gg Q_0, \quad \text{i.e.} \quad Q_2 \gg p_3 b^4. \tag{176}$$

Then we have, simply, that

$$R_5^{(i)} = \frac{b^2}{b^2-1}Q_2, \quad \text{i.e.} \quad R_1^{(i)} = \frac{b^2}{p_3(b^2-1)}Q_1 \tag{177}$$

which tend to

$$R_5^{(i)} = Q_2 \quad \text{or} \quad R_1^{(i)} = Q_1/p_3 \tag{178}$$

as $b^2 \to \infty$. Equation (177) does not disagree with Danielson's (1961 *b*) result: he makes the approximation (176) and obtains an equation

$$R_1^{(i)} = \frac{b^2}{p_2(b^2-1)}(b^4 + p_1 Q_1). \tag{179}$$

But in deriving this equation he has already assumed that $b^4 \ll p_1 Q_1$ and so (179) is itself valid only to that approximation and therefore equivalent to (177). Now (177) is in itself insufficient to determine $R^{(i)}_{\mathrm{min.}}$, though we can improve the approximation to obtain

$$R^{(i)}_{5,\,\mathrm{min.}} = \frac{b^2}{b^2-1}[Q_2 + \{p_3(\tfrac{1}{2}Q_2)^4\}^{\frac{1}{5}}] \tag{180}$$

with

$$b_i^{10} = \frac{Q_2}{2p_3} = \frac{1}{2}\frac{p_1^2}{p_2}Q_1. \tag{181}$$

Thus for $Q_1 \gg p_2^3/p_1^4$ we can take

$$R^{(i)}_{1,\,\text{min.}} = (p_1/p_2)\,Q_1. \tag{182}$$

However, as R_1 is increased beyond this value, other modes set in very rapidly and instability can occur for $b^2 = 2$ when $R = 2R^{(i)}_{1,\,\text{min.}}$. These graver modes prove to be much more efficient at transporting heat.

If we define

$$Q_3 = \frac{p_1^2}{p_2} Q_1 = \frac{\mu H^2 d^2}{4\pi^3 \rho \kappa^2} \tag{183}$$

so that Q_3—like R_3—is independent of both η and ν and varies only with κ, then for

$$(15/4p_3)^2 \ll Q_3 \ll 4$$

$$R^{(i)}_{3,\,\text{min.}} = 8Q_3^{\frac{1}{2}} \quad \text{with} \quad b_i^2 = 2, \tag{184}$$

while for

$$Q_3 \gg p_3^2$$

$$R^{(i)}_{3,\,\text{min.}} = Q_3 \quad \text{with} \quad b_i^{10} = \tfrac{1}{2}Q_3. \tag{185}$$

Now for $Q_1 \gg 1$, the exchange of stabilities can only occur when a temperature gradient

$$\beta_e = \frac{\pi \mu H^2}{4 g \alpha d^2} \frac{\kappa}{\eta} \tag{186}$$

is achieved. The exchange of stabilities then is *unaffected* by viscous effects but *hindered* by a high thermal conductivity (which makes the disturbance decay) and *assisted* by a high resistivity (which permits the lines of force to 'leak' through the fluid). On the other hand at the transition from overstable oscillations to steady instability, (184) and (185) are independent of both η and ν; therefore—although their ranges of validity are determined by η—the onset of instability is unaffected by viscosity or electrical resistivity. Moreover, when $Q_3 \gg p_3^2$, the critical temperature gradient

$$\beta_i = \pi \mu H^2 / 4 g \alpha d^2 \tag{187}$$

is independent of *all* the transport coefficients.

(*c*) *The nature of the motions*

We can draw a few conclusions about the nature of the convective motions. First, inspection of the perturbation equations shows that they fall into two independent groups: the first of these yields

$$\left[\left(\frac{\partial}{\partial t} - \eta \nabla^2\right)\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) - v_A^2 \frac{\partial^2}{\partial z^2}\right](\zeta, \xi) = 0 \tag{188}$$

for

$$\zeta = \operatorname{curl} \mathbf{u}\,.\,\mathbf{i}_z \quad \text{and} \quad \xi = \operatorname{curl} \mathbf{h}\,.\,\mathbf{i}_z$$

where the Alfvén velocity

$$v_A = (\mu H^2/4\pi\rho)^{\frac{1}{2}}. \tag{189}$$

The dissipative effects may be neglected near the onset of steady instability, leaving

$$\left(\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2}\right)(\zeta, \xi) = 0. \tag{190}$$

So this set of motions corresponds to Alfvén waves travelling along the field lines.

The remaining equations are similarly coupled and yield

$$\left(\frac{\partial^2}{\partial t^2}-v_A^2\frac{\partial^2}{\partial z^2}\right)w = g\alpha\frac{b^2-1}{b^2}\frac{\partial\theta}{\partial t} \tag{191}$$

in which the wave equation is modified by a source term. These two sets of motions each satisfy the continuity equation and are linked only by the boundary conditions.

Secondly, we can look at the time-dependence of the convective motions when $p_1 \ll p_2 \ll 1$. Chandrasekhar (1961) shows that overstability sets in with the frequency

$$\mu = \pi v_A/d \tag{192}$$

for $Q_3 \gg 1$; this result is actually valid over the wider range $Q_1 \gg b^4/p_2$. For non-oscillatory motions with $Q_3 \ll 1$ the time constant

$$\lambda = \pi v_A/d \tag{193}$$

but for $Q_2 \gg 1$

$$\lambda = \pi^2\kappa d^{-2}(Q_3 b_2)^{\frac{1}{3}} \tag{194}$$

and at the initial onset of instability

$$\lambda = \pi^2\kappa d^{-2}Q_3^{\frac{2}{3}}. \tag{195}$$

These results resemble those obtained for rotation; the characteristic time for an Alfvén wave to cross the layer determines μ when $Q_3 \gg 1$ and λ when $Q_3 \ll 1$, while thermal conductivity is unimportant unless $Q_3 \gg 1$ and only affects λ.

(*d*) *Conclusion*

The inhibition of convection by a magnetic field is more difficult to describe than that by rotation, since the equations are complicated by the presence of two Prandtl numbers. If (142) is assumed to apply, the equations are simpler and various results can be obtained, although it has not been possible to establish with precision the range of Prandtl numbers over which $R^{(i)}_{\text{min.}} < R^{(e)}_{\text{min.}}$ and the transition to steady instability is important.

The discussion in § 3 of the significance of the results for rotation might also apply here. In addition, we are now able to resolve a paradox: intuitive arguments indicate that convection will be halted if the kinetic energy of the motion is less than the magnetic field energy. But, according to Chandrasekhar's results, $R^{(e)}_{\text{min.}}$ is not reached until the available kinetic energy is far greater than the magnetic energy density. In particular, photospheric magnetic fields should be able to prevent steady convection, which they do not. It will be shown in part II that $R^{(i)}_{\text{min.}}$ *does* correspond to equipartition of energy between the velocity and the magnetic fields and the transition therefore assumes great importance. Moreover, as Danielson has argued, this criterion agrees with the observed suppression of convection by the magnetic fields of sunspots.

II. THE ONSET OF CONVECTION: A PHYSICAL APPROACH

5. Introduction

In the first part of this paper Danielson's (1961 *b*) results for the onset of non-oscillatory convective instability in the presence of a magnetic field have been generalized and similar criteria have also been obtained for rotating systems. It is obviously desirable to extend the treatment—if possible—to the non-linear heat transport problem. However, mathematical weapons seem to fail us here and we are reduced to physical arguments. A physical approach to non-linear convection should be applicable to the instability problem too and this part is devoted to the latter topic. Such an approach cannot pretend to great precision (it is accurate to within less than an order of magnitude) but it does enable us to understand the nature of the relevant phenomena and comparison with rigorously obtained results indicates that it is not misleading.

Before making any of these approximations, let us consider a single mode disturbance in a horizontal fluid layer of depth d, confined between 'free' boundaries, rotating with angular velocity Ω about the vertical z axis and heated below so as to give a superadiabatic temperature gradient β. Let the velocity and temperature disturbances be $\mathbf{u}$ and ϑ respectively and vary in the vertical direction as $\sin(\pi z/d)$. Then the linearized equations of motion are

$$\partial\vartheta/\partial t = w\beta + \kappa\nabla^2\vartheta, \tag{196}$$

$$\partial\zeta/\partial t = 2\Omega\,\partial w/\partial z + \nu\nabla^2\zeta \tag{197}$$

and

$$\frac{\partial}{\partial t}(\nabla^2 w) = -2\Omega\frac{\partial\zeta}{\partial z} + g\alpha\nabla_1^2\vartheta + \nu\nabla^4 w, \tag{198}$$

where α is the coefficient of thermal expansion, κ is the thermometric conductivity, ν the kinematic viscosity and

$$w = \mathbf{u}\,.\,\mathbf{i}_z, \quad \zeta = \operatorname{curl}\mathbf{u}\,.\,\mathbf{i}_z, \tag{199}$$

where $\mathbf{i}_z$ is a unit vector parallel to the z axis. The two-dimensional Laplacian

$$\nabla_1^2 = \nabla^2 - \partial^2/\partial z^2$$

and we can write

$$\nabla^2 = -\pi^2 d^{-2} b^2. \tag{200}$$

At the exchange of stabilities the left-hand sides of the equations vanish, so that from (196) and (197)

$$\vartheta = w\beta\tau_\kappa \tag{201}$$

and

$$\zeta = 2\pi\Omega w\tau_\nu/d, \tag{202}$$

where

$$\tau_\kappa = d^2/\pi^2 b^2\kappa \quad \text{and} \quad \tau_\nu = d^2/\pi^2 b^2 \nu \tag{203}$$

are the characteristic times for relaxation under thermal conductivity and viscosity respectively. Substitution from (201) and (202) into (198) then yields for the exchange of stabilities

$$R_1^{(e)} = (b^6 + T_1)/(b^2 - 1) \tag{204}$$

which is identical with the criterion originally derived by Chandrasekhar. If we were able to substitute a known eigenvalue for $\partial/\partial t$, we could similarly obtain criteria for overstability and the transition to steady instability but in general this is not possible.

Nevertheless, this approach does help to display the physical significance of the equations: from equation (196) we obtain a relation between w and ϑ depending on the shape of the cell through τ_κ which is inversely proportional to b^2; equation (197) gives the magnitude of ζ (and so of the horizontal eddy motions) in terms of w and τ_ν—or, alternatively, will fix τ_ν and so b^2 if we can decide ζ otherwise. Finally, equation (198) relates the size of the disturbance to the driving force: in this equation we are effectively balancing the work done (loss of potential energy) in the term $g\alpha\nabla_1^2\vartheta$ against the dissipation involved in the other two terms. We might, in fact, balance these quite simply in a non-rotating system where, for instability, the equation of motion gives

$$g\alpha\vartheta > w/\tau_\nu.$$

But from (196)

$$w\beta\tau_\kappa > \vartheta$$

so that if we put

$$\gamma = g\alpha\beta \tag{205}$$

we obtain

$$\gamma\tau_\kappa\tau_\nu > 1 \quad \text{or} \quad R_1 > b^4. \tag{206}$$

It is this general method of balancing the operative forces, or work done by them, that we shall develop in subsequent sections. (As Chandrasekhar has observed, a general variational principle can be established for these problems and such a principle corresponds to a balance between the work done by gravity and the dissipation in the system.) Comparison of (206) with (204) shows that they differ by a factor of $b^2/(b^2-1)$; this error is typical of the degree of approximation involved.

This method is easily adapted to the non-linear problem, when non-linear transfer of energy to higher modes, or inertial dissipation, can be substituted for viscous losses. However, it is reasonable to see first whether it can be applied to the linear instability problem, where comparison with analytical treatment is possible. This will be attempted here.

6. The effects of rotation and the Coriolis force

We have first to understand the general effect of rotation on convection and how it promotes stability. The rotational field does not absorb energy and it is only through the conservation of angular momentum that the motions are affected. This manifests itself in the equation of motion as the Coriolis force and it is this force that constrains the convective velocities to follow a more complicated pattern in attenuated cells. These eddies then enhance the viscous and inertial dissipation and it is only thus—indirectly—that convection is inhibited.

The Coriolis force $2\mathbf{\Omega}\wedge\mathbf{u}$ acts at right angles to the velocity and so can do no work; moreover, if $\mathbf{\Omega}$ is vertical only horizontal motions are affected. The rotation changes their direction without altering their speed or kinetic energy. Consider for the moment an isolated element of fluid moving horizontally in the rotating system with a speed u: then conservation of angular momentum compels it to lag behind the rotating fluid as it moves away from the axis of rotation, whereupon the ambient pressure gradient will force it inwards. Ultimately it returns to its place of origin. This is the effect of the Coriolis force and, if it alone acted, the fluid element would describe a circular path of radius

$$a_0 = u/2\Omega \tag{207}$$

with an angular frequency 2Ω.

Now let us combine this with a simple model of a convection cell. We know that such cells are polygonal and we do not harm the argument by restricting our attention to a cell that is square in cross-section. Suppose, then, that there is a rising current in the centre achieving a maximum vertical velocity w, and that near the upper surface of the cell the fluid abruptly streams sideways towards the corners of the cell, where it sinks again. The effect of rotation is to impose on this circulation eddies with opposite senses of rotation about the regions of rising and sinking material. The maximum radius a_0 of these eddies will be determined as in equation (207) above and the half-width of the cell is therefore given by

$$a = 2a_0 = u/\Omega. \tag{208}$$

Let us now assume that $u = w$. It was shown in part I that this is true at the onset of convection; furthermore, the discussion of non-linear convection in part III leads to the conclusion that the horizontal and vertical components of the velocity remain comparable there also. Now a can be written in terms of the quantity b, originally defined by equations (28), (200). Thus we have here that

$$a^2 = 2d^2/(b^2-1). \tag{209}$$

From (208) and (209), then, we obtain the useful relation

$$w^2 = u^2 = 2\Omega^2 d^2/(b^2-1) \tag{210}$$

which provides a quantitative measure of the effect of rotation on the convection cell. Thus the rotation, acting through the Coriolis force, drastically reduces the cross-section of the cell in accordance with equation (208); but the smaller a is, the smaller the dissipation times τ_κ and τ_ν become, and so the harder it becomes to maintain convection. So, although the Coriolis force can do no work itself and is unable to counteract the buoyancy forces, yet it forces the diminution of the horizontal dimensions of a cell and induces complicated whirling motions in it. It is the consequently enhanced dissipation that promotes stability; in a non-linear treatment the transfer of energy to smaller-scale modes would be increased for the same reasons and so heat transport would be impeded. We may note a fundamental difference here between the effects of rotation and of a magnetic field (despite the resemblance between the characteristic equations) for the magnetic field is able to do work on, or to absorb energy from, the velocity field. We must therefore expect its non-linear behaviour (at least) to be very different.

To be sure, in an actual cell of the largest permissible dimensions the motions would be rather more complicated than those described above. The horizontal velocities would not be confined to a thin boundary layer near the upper and lower surfaces but would occur throughout, though their maximum amplitude might be expected near the boundaries. The trajectory of an individual fluid element would then be more nearly a spiral about the rising current, as depicted by Veronis (1959). Nevertheless, the treatment above suffices for us to understand the problem.

7. Rotation and the onset of instability

From the discussion in the previous section there emerges an approach to the instability problem. Suppose that a certain size of cell is established and that a fluid element, starting with a small but finite velocity, traverses it from bottom to top. In a linear system the energy fed into a given mode remains in that mode (apart from viscous losses) so that the kinetic energy at the upper surface will equal the work done by the floating forces, less the dissipation. As the element nears the surface, the direction (but not the magnitude) of its velocity is altered and it indulges in the eddy motions described above.

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If its speed w then satisfies
$$w^2 > 2\Omega^2 d^2/(b^2-1) \tag{211}$$
it will be able to traverse the cell and a steady convective motion can be maintained. On the other hand, if (211) is not satisfied the element will be repelled by the Coriolis effects and so will retrace downwards the path that it had followed upwards; oscillations may then follow for a sufficiently high thermal conductivity and if the potential energy made available exceeds the viscous losses overstability will ensue. The resultant pattern of motions, once established, then persists, with ever-increasing amplitude.

If the Prandtl Number $p = \nu/\kappa$ is large, such oscillations cannot take place. Equation (210) then defines the width of cell attainable with a given w and the motions will persist only if the viscous dissipation in a cell of this size is sufficiently small, when convection sets in at the exchange of stabilities.

For a more quantitative treatment, the problem of the onset of non-oscillatory convection may be expressed thus: to find the condition that a fluid element, in its first transit through the layer, acquires enough kinetic energy for it to satisfy the inequality (211). We may distinguish two cases:

(i) $p \ll 1$ so viscosity is negligible and we are concerned with the transition from overstable to steady motions, and

(ii) $p \gtrsim 1$ so that viscous effects are important, oscillations are prohibited and we must find the condition for the exchange of stabilities.

Into this approximate treatment we introduce two concepts which must modify the agreement with the detailed analysis of part I. Their justification requires a study of the non-linear system, which is carried out in part III of this paper. First, we assume that $u \approx w$ so that results cannot be compared beyond the initial onset of instability. Secondly, balancing the dominant terms in (196) leads to the relation
$$\vartheta \sim w\beta\tau_\kappa. \tag{212}$$
The velocity then follows by taking the typical upward acceleration as
$$g\alpha\vartheta - w/\tau_\nu$$
where
$$w \sim (\gamma\tau_\kappa d - d/\tau_\nu).$$
This is satisfactory provided that $\vartheta < \frac{1}{2}\beta d$, the maximum average temperature excess that can be achieved in the layer, i.e.
$$w < d/\tau_\kappa \quad \text{or} \quad R_3 < 2b^4, \tag{213}$$
where
$$R_3 = pR_1 = \gamma d^4/\pi^4\kappa^2, \tag{214}$$
is a necessary condition for (212) to be applicable. When (213) is not satisfied, we must follow the well-known approach of mixing-length theory and put
$$\vartheta \sim \tfrac{1}{2}\beta d \tag{215}$$
whence
$$w^2 \sim (\tfrac{1}{2}\gamma d^2 - wd/\tau_\nu). \tag{216}$$
But if the motions across the cell are to persist at all, we must ensure that
$$w > d/\tau_\nu \tag{217}$$
in which case we can put
$$w^2 \sim \gamma d^2. \tag{218}$$

In so far as the approach in the next few paragraphs involves unrealistic assumptions of linearity it is not suggested that the criteria deduced must be strictly valid but they do bear comparison with the conclusions drawn from the perturbation equations. This is an encouragement to apply similar arguments to turbulent convection.

(a) *The case when* $p \ll 1$

Then if (213) is satisfied we may take

$$w \sim \tfrac{1}{2}\gamma\tau_\kappa d \tag{219}$$

and combining (219) and (214) respectively with the inequality (211) we obtain

$$R_3^2 > 2b^4T_3/(b^2-1) \tag{220}$$

and $$2T_3 < b^4(b^2-1), \tag{221}$$

where the Taylor number $$T_3 = p^2T_1 = 4\Omega^2d^4/\pi^4\kappa^2, \tag{222}$$

as necessary conditions for instability to take place. For $T_3 \ll 1$, (221) is trivially satisfied and the first non-oscillatory mode appears at

$$R_{3,\,\text{min.}}^{(i)} \sim 2(2T_3)^{\frac{1}{2}} \quad \text{with} \quad b_i^2 = 2. \tag{223}$$

(The exact treatment of part I gave $b_i^2 = 3$ and $R_{3,\,\text{min.}}^{(i)} = 3(3T_3)^{\frac{1}{2}}$.)

If $T_3 \gg 1$ and $b^6 \geqslant 2T_3$, then we have

$$R_{3,\,\text{min.}}^{(i)} \sim (2T_3)^{\frac{2}{3}} \quad \text{with} \quad b_i^6 = 2T_3. \tag{224}$$

but if $b^6 < 2T_3$ we must take $$w^2 \sim \gamma d^2 \tag{225}$$

which, with (211), yields $$R_3 > 2T_3/(b^2-1). \tag{226}$$

Thus, for b^2 of order unity, $R_3 \sim T_3$, while instability makes its first appearance as in equation (224), which may be compared with the results obtained in part I: $R_{3,\,\text{min.}}^{(i)} \doteqdot 3{\cdot}5\,T_3^{\frac{2}{3}}$ and $b_i^6 = 2T_3$. The agreement between the approximate treatment here and the rigorous derivation is very satisfactory.

(b) *The case when* $p \gtrsim 1$

We suppose that when convection first sets in, at the exchange of stabilities, the velocity attained suffices to satisfy (211). It must also satisfy (217), which is the stronger condition if

$$\tfrac{1}{2}T_1 < b^4(b^2-1). \tag{227}$$

Moreover, the inequality (206) has to be satisfied before convection can occur at all. Hence the rotation has no effect if $T_1 \ll 1$, while, if $T_1 \gg 1$ and (227) is satisfied,

$$R_{1,\,\text{min.}}^{(e)} \sim (\tfrac{1}{2}T_1)^{\frac{2}{3}} \quad \text{with} \quad b_e^6 \sim \tfrac{1}{2}T_1. \tag{228}$$

This is in tolerable agreement with the proper result

$$R_{1,\,\text{min.}}^{(e)} = 3(\tfrac{1}{2}T_1)^{\frac{2}{3}}, \quad b_e^6 = \tfrac{1}{2}T_1.$$

When $b^6 < \frac{1}{2}T_1$ the solutions of part I demand eddy velocities greater than w so that their connexion with this treatment is severed. However, this discussion is already of only formal interest: for if $p \ll 1$ instability will already have set in as indicated in equations (223) and (224), while if $p \gtrsim 1$ we have $R_3 \gtrsim R_1 > b^4$, so that (211), (217) and (225) give

$$R_3 > p^2b^4 \quad \text{and} \quad R_3 > T_3/2(b^2-1). \tag{229}$$

Therefore, for $T_3 \gg 1$, instability should first set in at

$$R^{(e)}_{1,\text{min.}} \sim p(\tfrac{1}{2}T_1)^{\frac{2}{3}} \quad \text{with} \quad b_e^6 = \tfrac{1}{2}T_1. \tag{230}$$

(When $p \sim 1$, the criteria (228) and (230) may be virtually indistinguishable from each other or from (224).)

8. An alternative approach

The treatment of the last section can be extended to cope with turbulent convection but there is also a simpler, quasi-dimensional approach to the instability problem, with analogous methods for rotation and a magnetic field. In the absence of any dissipation, equations (197) and (198) yield

$$\frac{\partial^2 w}{\partial t^2}+\frac{4\Omega^2 d^2}{\pi^2 b^2}\frac{\partial^2 w}{\partial z^2}=g\alpha\frac{b^2-1}{b^2}\frac{\partial \vartheta}{\partial t},$$

whose solutions have a time-constant $\tau = b/2\Omega$: (231)

this is the characteristic time in which conservation of angular momentum will destroy the disturbance. The energy in the convection depends on the gravitational time constant

$$\tau_g = (\gamma\tau_\kappa)^{-1} = \pi^2 b^2 \kappa / g\alpha\beta d^2. \tag{232}$$

Criteria for the onset of different forms of convection can be expressed simply in terms of these characteristic times and of τ_κ and τ_ν.

(a) *Overstability*

Before any instability can occur, (206) must be satisfied and overstable oscillations only arise if the perturbed temperature distribution can disappear before the motions are reversed. This demands a Prandtl number small compared with unity. Then overstability sets in when

$$\tau_g = \tau_\nu \gg \tau_\kappa \leqslant \tau \tag{233}$$

and rotation has no significant effect on (196) unless $T_3 \gg 1$, when $R^{(o)}_{1,\text{min.}} \sim T_3^{\frac{2}{3}}$ with $b_o^6 \sim T_3$.

(b) *Transition to steady motions*

For oscillations to occur, $\tau_\kappa \leqslant \tau$ and they can develop into steady motions when

$$\tau_g = \tau. \tag{234}$$

Thus, if $T_3 \ll 1$, $\quad R^{(i)}_{3,\text{min.}} \sim T_3^{\frac{1}{3}}$ with $b_i^6 \sim 1$

while, if $T_3 \gg 1$, $\quad R^{(i)}_{3,\text{min.}} \sim T_3^{\frac{2}{3}}$ with $b_i^6 \sim T_3$.

(c) *The exchange of stabilities*

Instability sets in here if viscosity dominates thermal conductivity; (234) must still be satisfied and, instead of (233),

$$\tau_g = \tau_\nu \leqslant \tau. \tag{235}$$

So rotation has no effect when $T_1 < 1$ but for $T_1 \gg 1$, $R^{(e)}_{1,\text{min.}} \sim T_1^{\frac{2}{3}}$ and $b_e^6 \sim T_1$.

It may be noted that, although overstability is observed to set in as predicted, Goroff's experiments confirm an intuitive feeling that oscillations should be inefficient at transporting heat. Discussion of non-linear effects may be expected to show that overstable oscillations are largely damped by the inertial dissipation.

9. Magnetic fields and the onset of instability

A magnetic field also inhibits convection but the convective motions themselves differ from those caused by rotation; the non-linear effects are also different. In a highly conducting fluid matter may be regarded as frozen on to the lines of force: motions across the magnetic field can distort these lines but it is still meaningful to speak of them. Each fluid element, so to say, 'remembers' something of its previous history. Also, if the magnetic field is concentrated by the convective motion it will absorb energy from it; alternatively, the magnetic field can do work and so give rise to motions in the fluid.

Consider a perfectly conducting fluid in a vertical magnetic field H and suppose that the lines of force are fixed at the boundaries of the layer (so that the vertical component of the perturbation in the magnetic field vanishes at $z = 0, d$). Then, if a small motion disturbs the system it will distort the lines of force and they will take up energy from it. This distortion can be thought of as a magneto-hydrodynamic wave propagated along the lines of force: the wave will be reflected at the bounding surface and when it returns it will restore the energy to the motion reversing its momentum. So oscillations will be produced and the lines of force will behave like elastic strings; the period of oscillation will then be the time taken by an Alfvén wave to cross the cell. Thus no energy is permanently lost to the magnetic field and if the thermal conductivity is high enough for the initial temperature disturbance to have disappeared then overstability may ensue (Cowling 1957). Steady motions can only follow when the kinetic energy of the disturbance exceeds the energy in the magnetic field, so that the motion punches its way across the cell despite the opposition of the field lines. As such a motion continues, the lines of force will be grossly distorted and wound up until they may ultimately prevent the motions from continuing and so destroy the cell.

If the electrical resistivity is high, it may be possible for the magnetic perturbation to decay (through ohmic losses) before it has had time to oppose the disturbing velocities; any instability could then proceed unhindered. This is the condition for the exchange of stabilities to take place.

The characteristic time for an Alfvén wave to traverse the cell is

$$\tau = \frac{d}{\pi v_A} = \left(\frac{4\pi\rho d^2}{\pi^2 \mu H^2}\right)^{\frac{1}{2}} \tag{236}$$

(as was shown in part I). Using this we can derive conditions for the onset of different modes of instability.

(a) *Overstability*

Assume that $\kappa \gg \eta \gg \nu$, i.e. $\tau_\kappa \ll \tau_\eta \ll \tau_\nu$, where $\eta = (4\pi\mu\sigma)^{-1}$ is the resistivity,

$$\tau_\eta = d^2/\pi^2 b^2 \eta \tag{237}$$

and τ_κ, τ_ν are defined similarly by (203). Then if any disturbance is formed (206) must be satisfied and the oscillations will grow if the thermal perturbation has disappeared before the magnetic effects reverse the motions, so that the floating forces can augment the amplitude of the disturbance. Hence we require that

$$\tau_g = \tau_\nu \gg \tau_\kappa \leqslant \tau. \tag{238}$$

If we define the Chandrasekhar numbers Q_1 and Q_3 so that

$$Q_3 = \frac{\eta\nu}{\kappa^2} Q_1 = \frac{\mu H^2 d^2}{4\pi^3 \rho\kappa^2}, \tag{239}$$

then magnetic fields have no significant effect till $Q_3 \gg 1$, when overstability sets in at $R^{(o)}_{1,\,\text{min.}} \sim Q_3$.

(b) *Transition to steady motions*

This is possible when the kinetic energy after one transit exceeds the magnetic energy; this condition reduces to

$$w^2 > v_A^2. \tag{240}$$

A treatment like that of § 7 leads to

$$R^{(i)}_{3,\,\text{min.}} \sim b^2 Q_3^{\frac{1}{2}} \quad \text{if} \quad Q_3 \ll 1, \tag{241}$$

and to

$$R^{(i)}_{3,\,\text{min.}} \sim Q_3 \quad \text{for} \quad Q_3 \gg 1. \tag{242}$$

Alternatively, the dimensional approach gives

$$\tau_g = \tau \geqslant \tau_\kappa \tag{243}$$

whence follow (241) and (242).

(c) *The exchange of stabilities*

This applies if the ohmic losses predominate; then we must have

$$\tau_g = \tau_\eta \leqslant \tau. \tag{244}$$

Then magnetic fields affect the onset of convection when $Q_1 \gg 1$, in which case

$$R^{(e)}_{1,\,\text{min.}} \sim Q_1. \tag{245}$$

The non-linear assumptions introduced in § 7 indicate that for $Q_1 \gg \kappa/\nu$, $R^{(e)}_{1,\,\text{min.}} \sim \eta Q_1/\kappa$. This condition is independent of η, κ and ν.

For comparison, the precise results of part I give as $Q \to \infty$

$$R^{(o)}_{1,\,\text{min.}} = Q_3, \quad R^{(i)}_{3,\,\text{min.}} = Q_3, \quad R^{(e)}_{1,\,\text{min.}} = Q_1.$$

III. CONVECTIVE HEAT TRANSPORT

10. Introduction

In part II we were able to establish that a physical approach, judiciously followed, can lead to sensible results regarding the onset of instability and this encourages us to try a similar method in discussing non-linear heat transfer. This part then is devoted to an attempt to obtain relations between the convective energy transport, the superadiabatic temperature gradient and the convective velocities that will enable us to answer the question: what temperature gradient is needed for a known amount of energy to be carried by convection? The discussion will be restricted to the case when the Prandtl number ($p = \nu/\kappa$) is small, which is of interest in astrophysics. After a brief review of other experimental and theoretical investigations a new approach to convection is laid down and formulae are established for the heat transport in a non-rotating and then in a rotating system. In the latter it can be shown that overstable oscillations are unlikely to transport more than a trifling amount of heat and that large-scale convection is not established until after the onset of steady instabilities as determined in part I. An estimate can also be made of the effects of a magnetic field on convection and these are treated in the penultimate section of this paper.

It is not to be expected that experimental results from a laboratory can simply be extrapolated and applied to the solar convective zone. Nevertheless, a brief resumé of the available information may be useful. Beyond the onset of convection it is convenient to define the Nusselt number N as the ratio of the convective energy transport to the product of the thermal conductivity $k = c_p \rho \kappa$ and the superadiabatic temperature gradient β (i.e. to the superadiabatic conductive transport)

$$N = E_{\text{conv.}}/k\beta. \tag{246}$$

N can then be expressed as a function of $(k\beta)$ and the Rayleigh, Prandtl, Taylor or Chandrasekhar numbers.

In the laminar range immediately beyond the onset of convection, where $R_c < R < 2R_c$, $N \sim \frac{1}{2}(R - R_c)$ but this does not persist. Near the critical Rayleigh number R_c convection appears in polygonal cells and settles down to a steady régime of hexagons. At higher Rayleigh numbers, although the convecting layer at any instant is entirely made up of more or less hexagonal cells, their particular configurations change in a time of the order of the turnover period of an individual cell. At higher Rayleigh numbers still, even this irregular pattern ceases to exist and small-scale turbulence ensues.

Typical results are summarized by Jakob (1957) and Howarth (1953): in the laminar range

$$N \propto R^{\frac{1}{4}} \quad (10^4 < R/p < 4 \times 10^5),$$

while under turbulent conditions

$$N \propto R^{\frac{1}{3}} \quad (R/p \gg 4 \times 10^5).$$

It seems that various transitions occur in this range and that different experiments may detect different effects. Malkus (1954) claimed that he could detect discontinuities in the slope of an R–N curve corresponding to the onset of higher modes of instability as predicted by linear theory; and Goroff (1960) found an increase in convective transport in an overstable rotating system near the exchange of stabilities. Perhaps new modes can appear as expected from linear theory, even in a highly non-linear system; however, one must in general conclude that it is too dangerous to extrapolate from the bewildering array of observations.

For well-established convection the only successful theoretical approach has been in terms of Prandtl's mixing-length l. In order to apply this theory, l must somehow—arbitrarily—be fixed. The practice is to equate l to the depth of the convecting layer d and then to assume that an element moves adiabatically and that all the work done by the buoyancy forces goes into kinetic energy. Then the mean temperature excess or deficit over the average at that level is taken to be

$$\vartheta \sim \tfrac{1}{2}\beta d, \tag{247}$$

whence the convected energy per unit area is

$$E = \tfrac{1}{2} R_3^{\frac{1}{2}} \pi^2 \beta k, \tag{248}$$

where κ is the thermometric conductivity, α is the coefficient of thermal expansion,

$$R_3 = pR_1 = pR/\pi^4 = \gamma d^4/\pi^4\kappa^2 \tag{249}$$

and

$$\gamma = g\alpha\beta. \tag{250}$$

No one could expect so simple a theory to be precisely accurate but it is not totally inadequate.

For sufficiently high Rayleigh numbers these motions will be turbulent and the treatment may have to be modified. The theory of turbulence, despite the efforts of many brilliant and ingenious mathematicians, remains inadequate. The most successful approach has been that of Kolmogorov: he considered the turbulent energy as being derived from the largest eddies, which are driven by convection or a shear flow and unaffected by viscosity; this energy is then transferred to smaller and smaller eddies by the non-linear effects until it is ultimately dissipated by viscosity in the smallest scale motions. If w is the driven velocity of the largest eddies, of scale d, the transferred energy per unit mass per second can depend only on w and d and so must have the form

$$\epsilon \sim w^3/d. \tag{251}$$

Kolmogorov then derived the energy spectrum of the inertial subrange, but this is of no help for the large-scale eddies that predominate in laminar convection and also in the sun.

Another approach to turbulent convection is to expand the termperature and velocity fields in normal modes. Ledoux, Schwarzschild & Spiegel (1961) used eigenmodes of the linear equations and a Heisenberg transfer coefficient to obtain relations between the modes in a statistically steady state. For large Rayleigh numbers they obtained r.m.s. velocity and temperature fluctuations

$$w \approx \gamma d^3/4\pi^2\kappa, \quad \vartheta \approx \beta d^2 w/6{\cdot}1\pi^2\kappa \tag{252}$$

and a mean energy transfer per unit area per second of

$$E \approx 5c_p\rho w\vartheta \approx \tfrac{1}{20}R_3^2\pi^2\beta k. \tag{253}$$

It is difficult to estimate the range of validity of their treatment, though it is certainly invalid if $R_3 \gg 1$. Equations (248) and (253) each give the energy transfer E but they are incompatible. The simplified treatment in § 12 will show the conditions under which (248) or (253) may be applied.

11. The principles of convective transfer

The problem of turbulent convection is sufficiently difficult that it seems advisable to try a less ambitious approach in order to gain some physical understanding of convection. We shall not presume to seek more than approximate relations, based on reasonable physical arguments. (In applications to the sun there are enough other ill-determined parameters to make great accuracy unattainable in any case.)

Let us then consider what happens well beyond the onset of convection. The original investigations of Bénard and those of his successors, the study of the onset of instability and the observations of convection in the atmosphere and in the sun all indicate that it is essentially a cellular phenomenon. Wherever a fluid element rises through the layer, another must sink to take its place and these two motions will be coupled: we shall assume that convection occurs in cells unless, for some specific reason, this is impossible. It is reasonable to suppose that the size and shape of these cells will be such as to minimize the energy lost to higher modes by non-linear interactions and dissipated by viscosity, so that their width should be comparable with the layer depth (unless this is prevented by some imposed constraint). If the velocity of the gravest mode is dominant, then we should expect from the continuity equation that the half-width a would satisfy

$$a = 2d, \tag{254}$$

so that, writing (28) in the form

$$b^2 = 1 + 2d^2/a^2 \tag{255}$$

we have $b^2 = \frac{3}{2}$, as was shown in § 3 (f).

We shall also claim as a corollary that, even if the system is turbulent, almost all the energy is carried by the largest eddies and that turbulent transport by the smaller scale motions is relatively inefficient. These smaller eddies are fed by buoyancy forces and by non-linear interactions, of which the latter are dominant. But even in a fully turbulent system the energy transport in an eddy of scale λ varies as $(\lambda/d)^{\frac{4}{3}}$ and the total energy transport is less than four times that due to the largest eddies acting alone.

Then, by assuming that all the energy is carried by eddies of the same scale as the depth of the layer and that the temperature gradient itself and the work done by the buoyancy forces are devoted to driving these eddies, it is possible to obtain simple relationships between E and β.* However, these simple assumptions are not always valid, nor do deductions from them tally with the experimental results quoted in § 10. This discrepancy arises because we have considered neither the effects of the boundaries nor those of the transformation of kinetic energy into heat by viscosity in the smallest eddies. These will be treated below.

At the upper and lower boundaries of the layer the convected heat has to be extracted from the tangentially moving fluid, by either a thermal or an eddy conductivity. This resolves itself into two requirements:

(i) that the time constant for thermal decay across the horizontally moving stream be less than the time taken for it to traverse the cell, and

(ii) that the temperature gradient across this stream be sufficient for the requisite amount of heat to be transported.

Let us define F so that

$$E = c_p \rho F. \tag{256}$$

Then, in order to remove heat at the upper surface there will be a boundary layer of thickness x, across which there is a temperature difference $\Delta\Theta$, moving with a typical speed w: then

$$\frac{x^2}{\pi^2 \kappa} < \frac{d}{w} \quad \text{and} \quad \frac{\kappa \Delta\Theta}{x} > F \tag{257}$$

are necessary conditions, subject to

$$\Delta\Theta < \tfrac{1}{2}\beta d \quad \text{and} \quad x < \tfrac{1}{2}d. \tag{258}$$

* Unno (1961) has substituted eddy coefficients of viscosity and thermal conductivity into the critical Rayleigh number and obtains expressions similar to those in § 12 below, which are also compatible with the theory of Vitense (1953).

Thus, if F is very large or κ very small there must be thin boundary layers at the surfaces, across which most of the temperature gradient will act, leaving only a small temperature difference for the comparatively trivial task of driving the motions between the layers. Then the problem has become one of a boundary layer and large cells are irrelevant. If the velocity varies across one of these boundary layers, as must happen if one of the boundaries is fixed, the convection will further be complicated by the high dissipation rate owing to kinetic and eddy viscosity.

In experiments these effects will all become important, since at least one boundary is rigid; however, in the convective zone and photosphere of the sun (257) is easily satisfied and the cells may be regarded as having free boundaries. Moreover, most terrestrial fluids have p of order unity. So it need not be demanded that convection in the sun and stars should obey the $R^{\frac{1}{4}}$ or $R^{\frac{1}{3}}$ laws experimentally obtained.

The energy ϵ defined by (251) is ultimately liberated as heat by viscous dissipation in the eddies of scale $\lambda_0 \sim (\nu^3 d/w^3)^{\frac{1}{4}}$. These small-scale motions should be homogeneous and so this heat will be uniformly distributed. Convection cannot remove this energy at the same rate everywhere and the driving temperature distribution will consequently be affected. Once again, the nature of the cells will be altered. First, an upper limit is set to the convective velocities, since the motions cannot be maintained if ϵ is too great, i.e. if

$$\rho w^3 > E. \tag{259}$$

So, for a fixed E, β must correspondingly be increased (though the dissipated energy should be included in the convective transport). Secondly, the uniform liberation of energy limits the lifetime of individual cells, otherwise heat could never be transported from regions where convection is insignificant. This may explain the laboratory result—apparently true also of the photospheric granulation—that the lifetime of an individual cell in a quasi-turbulent régime is only of the order of the turnover time in the layer.

12. Convection in the absence of rotation

We can now discuss the relatively simple problem of convection between free boundaries with $p \ll 1$. We shall assume that convection is well established, so that the Rayleigh number $R \gg R_c$ and w is large enough for the Reynolds number to be much greater than unity.

The amplitude of convection is then controlled by the inertial loss of energy to smaller eddies and we can assume that there is an inertial subrange of homogeneous eddies through which this energy passes before it is transformed into heat by the viscosity. The rate of change of energy in the whole system is obtained by integrating the scalar product of the equation of motion with the velocity over the whole volume. If we retain the Boussinesq approximation in a steady state we have

$$\int \mathbf{u}.[(\mathbf{u}.\nabla)\,\mathbf{u}]\,d\tau + \int \mathbf{u}.\nabla\left(\frac{p}{\rho}\right)d\tau = \int g\alpha\vartheta u_z\,d\tau + \int \nu\mathbf{u}.\nabla^2\mathbf{u}\,d\tau. \tag{260}$$

Since $\operatorname{div}\mathbf{u} = 0$, for any scalar ψ, $\int \mathbf{u}.\nabla\psi\,d\tau = 0$ and also $\int \mathbf{u}.[(\mathbf{u}.\nabla)\,\mathbf{u}]\,d\tau = 0$. Hence the left-hand side of (260) vanishes and the steady state is achieved when the rate of work done by the gravitational forces is balanced by the viscous dissipation. But we have postulated that the latter is determined by the inertial losses from the largest eddies and is given by (251). Thus the non-linear dissipation in the largest eddies must be equated to the mean work done by the buoyancy forces in them and so

$$w^3/d \sim \tfrac{1}{2}g\alpha\vartheta w$$

or

$$w^2 \sim \tfrac{1}{2}g\alpha\vartheta d. \tag{261}$$

Then, if the thermal conductivity is high enough, we can take the temperature of a rising or falling element as corresponding to the distance traversed in the characteristic time for thermal diffusion, so the temperature perturbation

$$\vartheta \sim w\beta\tau_\kappa \tag{262}$$

(see § 7), where the thermal relaxation time

$$\tau_\kappa = \frac{d^2}{\pi^2 b^2 \kappa}. \tag{263}$$

Then (261) reduces to $$w \sim \tfrac{1}{2}\gamma\tau_\kappa d. \tag{264}$$

Now the mean energy transported is $$F = w\vartheta \tag{265}$$

and thus $$F \sim \tfrac{1}{4}R_3^2 . \pi^2\beta\kappa. \tag{266}$$

The other modes present will modify this motion, so that the streamlines run more nearly along the edges of the cell and turn more abruptly at its top. This treatment will suffice so long as w is small and κ large enough for ϑ to be determined by the decay time τ_κ as in (262). Therefore it can be applied only for

$$\vartheta \ll \tfrac{1}{2}\beta d \quad \text{or} \quad R_3 \ll 1. \tag{267}$$

When this is so, the convective heat transport will be much less than the energy carried by conduction.

Convection only becomes important when R_3 is of order unity. If it utterly dominates conduction and $R_3 \gg 1$, then the motions are so rapid that an element is transported from bottom to top of the layer (or vice versa) without its heat content being affected by conduction. So the mean temperature difference carried will be $\vartheta \sim \tfrac{1}{2}\beta d$, which is independent of the conductivity κ. The temperature gradient is distorted by the motions until it is steep in boundary layers at top and bottom, between which there is only a slight gradient β' satisfying

$$\beta'/\beta \sim d/2w\tau_\kappa.$$

Then conservation of energy gives $$w^2 \sim \tfrac{1}{2}\gamma d^2 \tag{268}$$

whence $$F \sim (\tfrac{1}{8}R_3)^{\frac{1}{2}}\,\pi^2\beta\kappa. \tag{269}$$

From (259) this treatment is adequate so long as

$$w^2 \ll c_p\vartheta \tag{270}$$

which, when $R_3 \gg 1$, reduces to

$$g\alpha d \ll c_p \quad \text{or, for a gas, to} \quad gd \ll c_p\Theta. \tag{271}$$

Thus (270) implies that the potential energy drop across the layer must be small compared with the thermal energy of the gas. Unless (271) is satisfied, we must take

$$w^3 = c_p F \tag{272}$$

whence $$F \sim [c_p(\tfrac{1}{2}\beta d)^3]^{\frac{1}{2}}. \tag{273}$$

The energy fluxes defined by (269) and (273) both vary as $\beta^{\frac{3}{2}}$ and are in the ratio $g\alpha d/c_p$, the lesser being applicable.*

We now suppose that (270) is fulfilled: then comparison of (266) and (269) with (253) and (248) shows that these results agree with those of Ledoux *et al.* (1961) when $R_3 \ll 1$ and with the mixing length approach for $R_3 \gg 1$. The treatment in this section demonstrates the transition from one régime to another when convection dominates conductive transport. The other eddies present in the turbulent motion will interfere with the fundamental so that the resultant motion closely corresponds to an element moving vertically through the layer and then abruptly travelling along the horizontal boundaries. This is the motion postulated in mixing-length theory, but it seems proper to suppose that the speeds are controlled by inertial dissipation rather than by the energy fed into an isolated element.

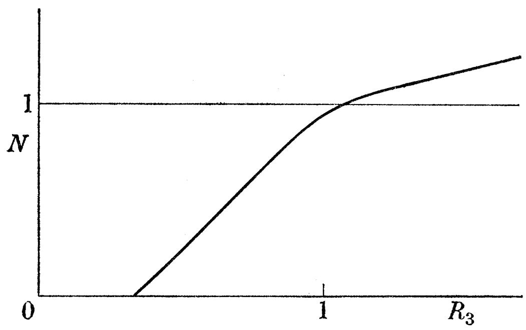

FIGURE 6. Convective transport in the absence of rotation ($T_3 \ll p^2$). $N = 1$ gives the conductive heat transport; for $R_3 \ll 1$, $N \propto R_3$ and for $R_3 \gg 1$, $N \propto R_3^{\frac{1}{3}}$.

Since both effects depend upon the non-linear terms in the equation of motion it is natural that they should give the same result; however, the approach presented here can be generalized to cover the narrow cells ($b^2 \gg 1$) produced by rotation or a magnetic field and therefore appears to be preferable. Moreover, the condition (270) and the alternative transport governed by (273) are not apparent from the mixing-length approach, nor is the transition to another régime when $R_3 < 1$.

The variation of the convective transport with Rayleigh number is illustrated by the logarithmic plot of the Nusselt number

$$N = F/\pi^2\beta\kappa \tag{274}$$

against R_3 in figure 6.

13. CONVECTION AND ROTATION

The efficacy of rotation in hindering convection can now be investigated. In § 6 it was shown that the effect of the Coriolis force is to decrease the width of the convection cells: this increases the inertial transfer of energy to smaller eddies and so diminishes the heat transport. As in the preceding section, the work done by the gravitational forces must be equated to the non-linear losses in order to find a statistically steady state for the dominant eddies, since the Coriolis forces—like the pressure gradients—do no work. This discussion is complicated by the occurrence of overstability before steady convection can set in; it will

* Unsöld (1955) asserts that mixing-length theory is valid provided that $w \ll c$, the local sound velocity. But $c^2 \sim c_p\Theta$, where Θ is the absolute temperature and so (270) is a far more restrictive condition unless ϑ and Θ are comparable.

be indicated in §§ 14 and 15 that oscillations are relatively inefficient at transporting heat and criteria will then be derived for the maintenance of large-scale motions.

However, it is necessary first to distinguish between the eigenmodes of the linear equations, which determine the motions at the onset of instability, and the eddies—which may loosely be referred to as turbulent 'modes'—that will be present well beyond this point. We saw in the last section that if only one linear mode existed there could be no loss of energy from it through the $(\mathbf{u}\,.\,\nabla)\,\mathbf{u}$ term. In a rotating system there is an induced horizontal, vortical velocity $\mathbf{u}_2$ (with amplitude u) in addition to the vertical velocity w and the normal horizontal circulation $\mathbf{u}_1$. Moreover, it can be shown that for the linear modes $\mathbf{u}_2\,.\,\nabla w = 0$ and there exists some scalar ψ such that $(\mathbf{u}_2\,.\,\nabla)\,\mathbf{u}_2 = \nabla\psi$. So, although the non-linear term builds up other modes from the energy of the fundamental, they might be restricted so that the inertial transfer rate from a linear mode in an elongated cell would be only $(u^2+w^2)\,w/d$ and independent of the cell-width. In that case we should have

$$u^2+w^2 \sim g\alpha\vartheta d, \tag{275}$$

while the curl of the equation of motion yields the relation

$$u^2 = 2\Omega^2 d^2/(b^2-1) \tag{276}$$

between u and the angular velocity Ω. The same result was derived, from physical arguments, in § 6.

If $u \geqslant w$, these equations then give almost the same conditions for the maintenance of steady convection as were obtained in parts I and II for the transition from oscillatory to steady instabilities, namely

$$R_3 > b^2\, T_3^{\frac{1}{2}}/(b^2-1)^{\frac{1}{2}} \quad \text{if} \quad T_3 \ll b^4(b^2-1) \tag{277}$$

or

$$R_3 > T_3/(b_2-1) \quad \text{if} \quad T_3 \gg b^4(b^2-1), \tag{278}$$

where

$$T_3 = p^2 T_1 = 4\Omega^2 d^4/\pi^4\kappa^2.$$

The analysis of § 3 showed that linear modes beyond the initial onset of convection have horizontal eddy velocities u that can be much greater than the vertical velocity w. In a non-linear system, on the other hand, we should expect to have $u \sim w$. A linear treatment imposes the condition that all perturbed quantities exhibit a similar dependence on space and time for a given mode, whence came the result that

$$u = T_3^{\frac{1}{2}}\, w/b^2 \quad (T_3 \gg 1) \tag{279}$$

but non-linearity frees us from this restriction; moreover, the energy of such rapid motions could only be supplied

(i) by work done by the pressure gradients, or

(ii) by advected energy from the vertical motion.

That pressure gradients should act along the eddy velocities and be strong enough to drive them against inertial dissipation is difficult to conceive, for the pressure itself can only be maintained by the flow of fluid from lower levels. The advected kinetic energy, on the other hand, can only be that corresponding to a velocity w; so that once the motion is established the eddies must relax until u is comparable with w. Again, if u were determined by (279) for all Rayleigh numbers, then the convective transport would be inversely

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proportional to $T_3^{\frac{1}{3}}$ (for $T_3 \gg 1$) however great R_3 might be. But it is apparent that as the velocities become infinite the effects of rotation must degenerate to no more than a trifling perturbation. We shall therefore assume that

$$u \lesssim w. \tag{280}$$

When turbulence is fully established, the energy transfer to the smallest eddies cannot remain independent of the width of the cells, so when we equate this loss to the work done by the floating forces on the dominant 'mode' we must replace equation (275) by the stronger condition that

$$b(u^2+w^2) \sim g\alpha\vartheta d. \tag{281}$$

The cell-width is restricted by the Coriolis forces but the arguments advanced in § 6 indicate that the value of b is still given by (276). This attenuation of the cells enhances the inertial dissipation in (281) and so renders the convection more difficult to maintain. Hence convection must occur in elongated cells until the Rayleigh number is large enough to make

$$w^2 \gg 2\Omega^2 d^2. \tag{282}$$

If this inequality is satisfied, b^2 is of order unity, the cells are more or less equidimensional and the vortical velocity drops to some value, much less than that of w, which will satisfy (276). In this limit, the rotation can have no more effect in restraining the convective transport, which can then be determined as in § 12.

14. Overstability and heat transport

If $p \ll 1$, instability first appears, as overstable oscillations, when $R_3 = 13{\cdot}5p$ if $T_3 \ll 1$ or when $R_3 = 6p(\frac{1}{2}T_3)^{\frac{2}{3}}$ if $T_3 \gg 1$. The normal modes of the linearized equations show that the simple circulation $(\mathbf{u}_1 + w\mathbf{i}_z)$ and the vortical motion $\mathbf{u}_2$ have the same amplitude but are 90° out of phase with each other. Kinetic energy is therefore transferred from one set of motions to the other without being dissipated and the angular frequency of oscillations at the onset of a given mode is $\mu = 2\Omega/b$.

If the oscillations build up to large amplitudes the inertial terms will once again become important and the non-linear eddies will differ from the linear eigenmodes. Turbulent oscillatory convection would take the form of a large-scale abortive circulatory motion which is prevented from continuing by the Coriolis forces. However, it is difficult to see how the kinetic energy could be stored in the vortical motions (unless the initial disturbance is limited to a narrow region, in which case the inertial dissipation would obliterate it) and the energy acquired in each oscillation would probably be lost through turbulence to the smallest eddies where it could be dissipated by viscosity. If a fluid element achieved a velocity of magnitude w in the course of an oscillation, it might be expected to spread through a distance

$$a \sim w/2\Omega \tag{283}$$

and the angular frequency of the motions would then be of order 2Ω.

An overstable perturbation might grow into two forms of large-scale motion: either convection could occur as oscillations of the sort just described or else the disturbance might achieve a finite velocity large enough to satisfy (276) and so develop into a steady (non-oscillatory) motion. If the latter occurs, there must be sufficient energy available from the floating forces to balance the inertial dissipation; and we saw in the previous section that this cannot be so, even for linear eigenmodes, until *after* the rotating system is unstable to non-oscillatory perturbations. So no steady convection is possible until the instability criteria (277) and (278) are satisfied—and turbulent motions will be still further restricted by equation (281).

Heat could still be transferred by oscillatory motions, though it is unlikely that these could ever be an efficient means of transport. If we consider circulation in a large O-shaped eddy between the bounding surfaces, when T_3 is large, a steady motion (if it were possible) would bring material from the bottom to the top of the layer, while oscillations can have only a small amplitude a and merely carry fluid from a distance a below the upper surface. So the energy transported by oscillatory modes must be less by a factor of $(a/d)^2$.

If $T_3 \ll 1$ and $R_3 \ll 1$ convection is controlled by the thermal decay time τ_κ and so overstable and steady motions might be of comparable efficiency, though neither would be significant compared with the conduction of heat. Moreover, the non-linear terms introduce harmonics which might be expected to interfere with the fundamental frequencies and so to impair the efficiency of transport. Thus it is likely that turbulent oscillatory convection, even if it is possible, is always far less efficient than a steady motion of the same amplitude might be.

For a given Rayleigh number, larger cell sizes are possible for oscillations than for steady motions. Thus the constraint imposed by the non-linear effects would be relaxed. However, the non-linear oscillations described earlier in this section cannot be maintained unless the system is already unstable to steady perturbations too. For the energy loss at the end of each bout of movement is of order $2\Omega w^2$ per unit volume and must be balanced by $g\alpha\vartheta w$, the work done. If oscillations of this nature occur at all, the inequality

$$g\alpha\vartheta > 2\Omega w \tag{284}$$

must therefore be satisfied. If $T_3 \ll 1$, (284) requires that $R_3 \gtrsim T_3^{\frac{1}{3}}$ while for $T_3 \gg 1$ it reduces to $R_3 \gtrsim T_3$. These conditions are equivalent to—or stronger than—(277) and (278) so it is not possible to maintain these turbulent oscillations while the system remains stable to steady perturbations. Thereafter, oscillatory modes are not solutions of the linearized equations and will be unimportant in a non-linear system.

Thus we have shown that overstability is likely to give rise only to small convective motions, too slow to be turbulent; such disturbances cannot be expected to transport much heat and therefore convection should only become significant beyond the onset of steady instabilities. This is confirmed by Goroff's experiments, which show that the Nusselt number has a value of only 1·06 through the overstable range. The nature of convection beyond this region is treated in the following section.

15. Heat transport in the presence of rotation

If large-scale turbulent convection takes place, equations (276), (280) and (281) must be satisfied, whence we find that for $T_3 \ll b^6$ convection is possible for

$$R_3 \gtrsim b^3 T_3^{\frac{1}{3}}/[2(b^2-1)]^{\frac{1}{2}} \tag{285}$$

and can first be maintained with

$$b^2 = \tfrac{3}{2}, \quad R_3 \sim \tfrac{27}{8} T_3^{\frac{1}{3}}. \tag{286}$$

Beyond this value of the Rayleigh number the convected energy is determined by equation (266). If, on the other hand, $T_3 \gg b^6$, turbulent convection can be maintained when

$$R_3 \gtrsim \frac{2T_3}{b} \tag{287}$$

and first becomes possible at $$R_3 \sim 2T_3^{\frac{5}{6}}, \tag{288}$$

above which value the largest cell possible has $b \sim 2T_3/R_3$ so that the heat transport is given by $$F \sim R_3 \pi^2 \beta\kappa / 4T_3^{\frac{1}{2}}. \tag{289}$$

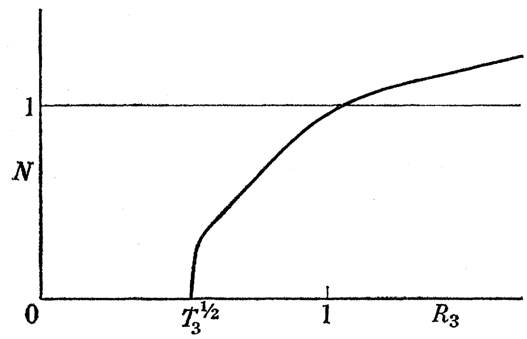

FIGURE 7. Convective transport in the presence of rotation ($p^2 \ll T_3 \ll 1$). $N = 1$ gives the conductive transport; for $T_3^{\frac{1}{2}} \ll R_3 \ll 1$, $N \propto R_3^2$ and for $R_3 \gg 1$, $N \propto R_3^{\frac{1}{2}}$.

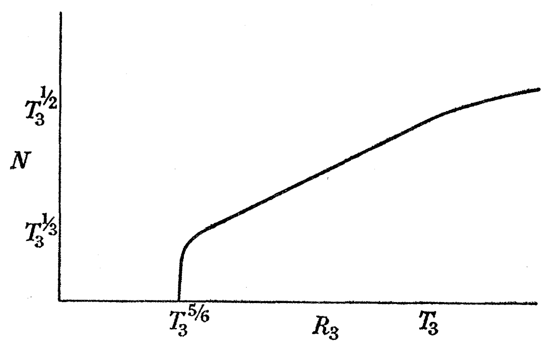

FIGURE 8. Convective transport in the presence of rotation ($T_3 \gg 1$). For $T_3^{\frac{5}{6}} < R_3 < T_3$, $N \propto R_3$ and for $R_3 \gg T_3$, $N \propto R_3^{\frac{1}{3}}$.

Once $R_3 \gg 2T_3$, rotation ceases to affect the convection and the heat transported is determined by (269); however, in the range $2T_3^{\frac{5}{6}} \ll R_3 \ll T_3$, the heat transport from (289) is less by a factor of $(R_3/T_3)^{\frac{1}{2}}$ than that from (269).

The inertial energy loss cannot be higher than was assumed in deriving (285) and (289) and this discussion indicates that turbulent convection appears with a finite heat transport. The question immediately arises: what is the nature of the motions before R_3 exceeds the values set above? After steady instability becomes possible on the linear theory, small motions will appear but they will be restricted so as to keep the inertial energy losses small and it is unlikely that the heat transport would rise abruptly before the value of R_3 set by (286) or (288) is approached.

If a set of measurements were made of the Nusselt number as the Rayleigh number was increased, for $p^2 \ll T_3 \ll 1$, a logarithmic plot of N against R_3 might appear as in figure 7.

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For $R_3 \ll T_3^{\frac{1}{2}}$, convective transport is negligible, but in the neighbourhood of $R_3 \sim T_3^{\frac{1}{2}}$ convection increases suddenly in efficiency and thereafter the convective transport is the same as it would be in the absence of rotation. Since conduction dominates convection for $R_3 \ll 1$, we may say that rotation cannot appreciably affect heat transfer unless $T_3 \gtrsim 1$.

When $T_3 \gg 1$, N should vary with R_3 as shown in figure 8: convection sets in as overstable oscillations when $R_3 \sim p T_3^{\frac{2}{3}}$ but their effects are negligible. When $R_3 \gtrsim T_3^{\frac{2}{3}}$, steady instabilities become possible and convection can become important compared with conduction. However, the convective transport will probably only rise sharply near $R_3 \sim T_3^{\frac{5}{6}}$, whereafter N varies as R_3 until $R_3 \sim T_3$. The heat transport around $R_3 = 2T_3$ may be less by a factor of 2 or 3 than in the non-rotating case but once $R_3 \gg T_3$ the effects of rotation can be ignored.

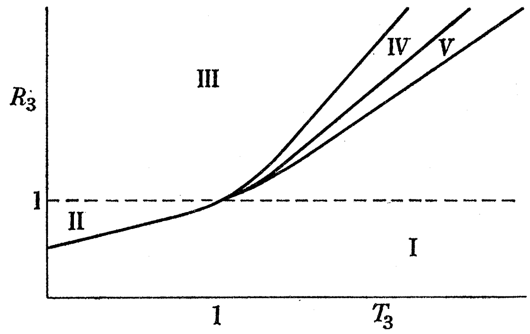

Figure 9. The effects of rotation on convective transport. For $T_3 \ll 1$, $R_3 \propto T_3^{\frac{1}{2}}$; when $T_3 \gg 1$, R_3 varies as $T_3^{\frac{2}{3}}$, $T_3^{\frac{5}{6}}$ and T_3. For the numbered regions, see table 5.

Table 5. The effects of rotation on convective transport

(referring to the numbered regions in figure 9)

I	Convection negligible and heat transport entirely by conduction
II	Convection occurs and is unaffected by rotation but heat transfer occurs almost entirely by conduction
III	Heat transfer dominated by convection, which is unaffected by the rotation
IV	Heat transfer dominated by convection but is decreased by the rotation
V	Heat transfer by convection is much less than that in IV but may be comparable with or larger than the conductive transfer

The effects of rotation on heat transfer are indicated in figure 9 and table 5. It may be noted that the exchange of stabilities first occurs when

$$R_3 = 3p^{-\frac{1}{3}}(\tfrac{1}{2}T_3)^{\frac{2}{3}} \tag{290}$$

and the largest modes have a state of marginal stability only when $R_3 \approx 2T_3/p$ so that when $p \ll 1$ convection becomes important—in regions IV and V of figure **9**—*before* the exchange of stabilities can take place.

Once $p > \frac{1}{3}$, steady convection sets in at the exchange of stabilities; for Prandtl numbers of order unity, large-scale motions will be possible at about the same Rayleigh number as the state of marginal stability, i.e. the transition from small-scale to turbulent motions is gradual.

The conclusions reached in this section may be summarized as follows:

(i) Rotation hinders the transport of heat by convection if $T_1 \gg 1$.

(ii) Overstable disturbances are of negligible importance in transporting heat.

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(iii) Convective transport is unimportant if $R_3 \ll 1$; but if $p^2 \ll T_3 \ll 1$ such convection as can occur is hindered until $R_3 \gtrsim T_3^{\frac{1}{2}}$, whereafter rotation has no effect.

(iv) If $p \ll 1$ and $T_3 \gg 1$, convection of heat is negligible compared with conduction throughout region I of figure 9 but may become appreciable in region V when $R_3 \gtrsim T_3^{\frac{2}{3}}$. The convective transport probably increases rapidly in the transition from V to IV when $R_3 \sim T_3^{\frac{5}{6}}$.

(v) When $R_3 > T_3$, rotation ceases to affect the heat transport. (This statement should remain valid even if the predictions in (iv) are not correct in detail.)

(vi) If $p \sim 1$, large-scale convection can be maintained near the exchange of stabilities.

16. Convection in a magnetic field

Similar results can be obtained for convection in the presence of a magnetic field. This problem is simpler, for the circulatory motions are quite straightforward and there are no vortices like those induced by Coriolis forces. A magnetic field differs from a rotation field in that it can absorb energy from the motion; in a highly conducting fluid, moreover, the material is 'frozen' on to the lines of force so that any motion must distort the magnetic field and work must be done in compressing the field lines before the circulation can cross the cell. This limits the occurrence of steady instability but the magnetic field can also react on the motions after instability has occurred; for the circulation will further distort the lines of force and in general increase the magnetic forces resisting motion.

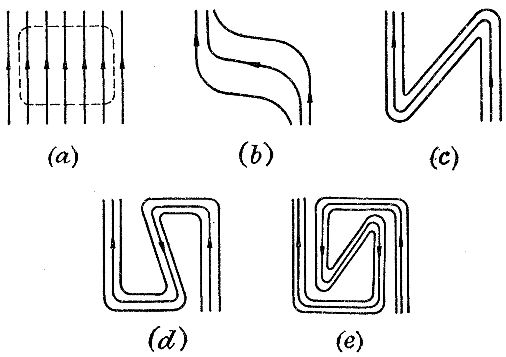

Figure 10. The distortion of a magnetic field by convection.

Let us consider what happens if steady convection begins well beyond the onset of instability, so that the motions dominate the magnetic field. Then the uniform field shown in figure 10 (*a*) will be distorted and concentrated as it is carried round by the circulation, so that the lines of force pass through the configurations depicted in figure 10 (*b*), (*c*), (*d*), (*e*), etc., and are gradually 'wound up' by the motions. However, given a finite resistivity, these lines of force can reconnect where the gradients are high and it is most likely that the convection will concentrate the field into 'ropes' of flux centred on the rising and falling currents but with no motions across them, and that these ropes will have no significant effect on the convection. The highly contorted fields within the cells might in general be expected ultimately to decay through ohmic losses. So the final field configuration will probably resemble that depicted in figure 11, with negligible fields except in restricted regions of high magnetic flux, where no motions can occur (so the field strength must be sufficient in these regions to prevent convection). In the remaining space convection proceeds, unhampered by any electromagnetic forces. This must be distinguished from the 'spaghetti' concept of turbulent magnetic fields, in which it is supposed that the turbulent motions will stretch out the field lines until equipartition is achieved between magnetic and kinetic energies (Batchelor 1950): such equipartition can only occur if the steady field in the turbulent region is supplied and maintained by external sources, whereas the configuration of figure 11 has been achieved precisely by removing the steady field from the turbulent regions and so effectively changing the boundary conditions. For the same reason Moffatt's (1961) magnetic energy spectrum cannot be applied to the turbulent region here, and the magnetic fields in it are free to decay.

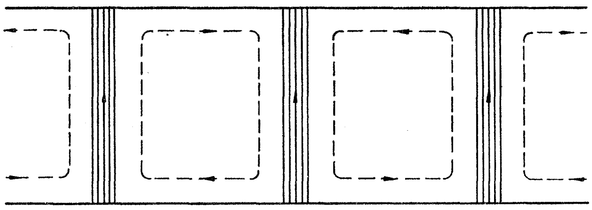

FIGURE 11. Convection in a magnetic field.

Now let us suppose that the Rayleigh number is decreased from the very high value that has been assumed so far, until it is only just sufficient to produce convection in a fluid with

$$\nu \ll \eta \ll \kappa, \tag{291}$$

where η is the resistivity and ν the kinematic viscosity. Then the convective motion in a cell will distort the field as in figure 10 but as it does so it is compelled to maintain the distortion against ever-increasing electromagnetic forces and there is no time for these to decay. So the buoyancy forces, which could only just maintain motions against the initial field, will be overcome by the enhanced magnetic forces, the motion will be halted and perhaps even reversed by the magnetic pressure gradients and the convective system will break down. The lifetime of the convective cell will therefore be of the order of the time taken for an element of fluid to circulate around it. We can determine when it is possible for an instability to develop into quasi-steady convection, of the type just described. For the effect of the motion is to compress the field, so producing a gradient of magnetic pressure, and to curve the lines of force. The curvature forces are negligible for a cell in which $b^2 \gg 1$ and for a cell of half-width a the magnetic force resisting motion is of the order $\mu H^2/2\pi a$; so the work done by a fluid element in overcoming this is $\mu H^2/2\pi$, which is less than the kinetic energy of the fluid element if

$$w^2 > \mu H^2/\pi\rho. \tag{292}$$

But, as before, the motions are limited by non-linear effects so that

$$g\alpha\vartheta d \sim bw^2$$

and steady convection can first be maintained with $b^2 \sim 1$ and

$$R_3 \sim Q_3^{\frac{1}{3}} \quad \text{if} \quad Q_3 \ll 1, \tag{293}$$

or

$$R_3 \sim Q_3 \quad \text{if} \quad Q_3 \gg 1, \tag{294}$$

where the Chandrasekhar number

$$Q_3 = \mu H^2 d^2 / 4\pi^3 \rho \kappa^2.$$

These are the same as the conditions derived from the linearized equations in part I for the onset of steady instabilities. Therefore we may once again say that overstable oscillations cannot develop into steady convection until after the system is unstable to steady perturbations in any case. Unless R_3 exceeds the value set by (293) or (294), the only convective transport must be by oscillatory motions.

Since the kinetic energy can be transformed into magnetic energy, turbulent oscillations (unlike those that arise in the presence of rotation) could be maintained although the arguments advanced in § 14 indicate that they will not be very important as a means of transporting heat.

Danielson (1961 *b*) showed that convection in the solar photosphere (where R_3 and Q_3 are less than unity) seems to be inhibited unless steady motions can occur, which supports the thesis that oscillatory convection is never a very efficient means of transporting heat.

17. Conclusion

In this part, reasonable assumptions were made as to the effects of rotation and a magnetic field on convection and we have succeeded in deriving thence some plausible and consistent estimates of the magnitude of the heat transport. The justification for such a procedure is that it suggests experiments that will confirm or reject the results obtained.

The only relevant experiments are those of Goroff (1961), using mercury in a rotating dishpan: his results are not inconsistent with the conclusions drawn in § 15. Further measurements, for larger values of T_3 and R_3, would be valuable.

The discussion showed that steady turbulent convection could not develop from overstable perturbations unless the system were already unstable to non-oscillatory modes. This follows from the analogy between the arguments advanced in part II and those in § 13. In part II it was claimed that instability could not follow unless the energy fed into a mode enabled a fluid element to cross the cell immediately; the energy given to such a Lagrangian element derives from the $(\mathbf{u} \cdot \nabla)\,\mathbf{u}$ term of the Eulerian equations and is therefore comparable with the energy passed through the inertial subrange for viscous dissipation (though the latter is the larger if $b^2 \gg 1$).

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